# PRINCIPLES OF NATURAL PHILOSOPHY

BY

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# PREFACE

NATURAL philosophy is the study of nature, or the universe. Experience shows that all effects are preceded by causes, and under identical circumstances all changes are always effected in the same manner. Nature is in a continual state of flux, this flux consisting of a ceaseless interchange of energy between its various parts, but its action is always uniform and unique. It is this invariableness and uniformity which renders possible a philosophy of nature. The investigation of the manner or laws under which changes take place is comprised under such names as Physics, Dynamics, Mechanics, or Natural Philosophy, but since in this book we shall not confine ourselves merely to the investigation of such processes, but shall consider ourselves free to speculate upon them, insofar as that is possible, the latter term is preferable. It was used by Newton in this sense.

The ultimate, or primordial causes of nature must ever remain unknown to us; since the infinite cannot be grasped by finite minds, but a certain proximate region has been explored and it is to the credit of mankind that its borders are being continually extended. No important result can ever be achieved without serious and concentrated thought and close and careful reasoning. Such a method, in any of its forms, constitutes mathematics, and any man who reasons carefully and accurately is a mathematician. The theory of changes was called by Newton, its discoverer, Fluxions, symbolizing thereby the fluxions of nature. It is now known as The Calculus, and it is only decent that every man with more than a rudimentary education should know the calculus and thereby something of the universe in which he lives. It is the ability and willingness to think which has raised civilized man

from the savage, and it is our knowledge of nature and natural forces which determines our present, or any future civilization.

The study of such a subject must react upon the student in forming within him a new realization of what truth actually is. He will find that many of his most cherished beliefs would not stand before the Court of Nature, or for that matter before an ordinary law court. For ages man has seemed incapable of distinguishing between the true and false, and has had little desire to do so even when the means were at hand. The idea prevailed, and still prevails, that truths could be created by authority. A statement if loudly proclaimed and accepted by a sufficient following was, and is, supposed to be true irrespective of any inherent quality. A man may say that he believes and accepts as a truth something which he does not understand, but unless he clearly recognizes for himself the reasons why a thing *must* be true, it is not a truth for him. The student of mathematics learns from the beginning that nothing but the truth can ultimately prevail and that what is false necessarily carries with itself its own annihilation. Authority has no place in Science, for its results do not rest, or need to rest, upon any personal sponsor, no matter how distinguished, but solely upon the truth or falsity of the reasoning by which they are derived.

Text books on this subject have been overfull of problems concerning rods and strings and flies walking on circular wires or perfectly smooth tables, so that it is perhaps natural that the impression has prevailed that instead of being an instrument for the acquisition of truth, mathematics are chiefly an agglomeration of symbols by which fantastic results, having no human or practical interest, are obtained. But natural philosophy is not the study of rods and strings. It is the story of the universe. For this reason we shall confine ourselves to processes which are actually occurring about us all the time — to our earth, and to our solar system.

# PRINCIPLES OF NATURAL PHILOSOPHY

# NATURAL PHILOSOPHY

## 1. The Fundamental Law

THE first efforts of man to understand the universe were by metaphysical processes. Without examining the universe as it is, they sought to evolve its plan out of their inner consciences. They conceived in their minds a universe as they thought it might be, or could be, or should be, and sometimes by *deduction* sought to fit their theory to the case. Generally they did not trouble themselves to go so far as this, but rested content with their imaginary universe. The method consisted of assuming causes instead of interrogating nature herself. They further exhibited the strange tendency of mankind to imagine that by the use of sufficient meaningless words, definite thoughts could be expressed. There were necessarily as many systems as philosophers. But truth is unique, so that only one of all these systems could possibly be true, while the *a priori* probability was that all were false. In the middle ages when there was no science (knowledge) it was natural that men should have exercised their minds with "Beating the air," but that such methods should persist to the present day is an anachronism. Sooner or later these relics of mediaevalism must disappear. Among certain workers on the present borderland there is a tendency to revert to these unsound methods, and within recent years a new kind of metaphysical physics has been developed. The student of natural philosophy must carefully avoid the unclean thing.

We recognize that the universe consists of matter although we do not know what matter is, or when or how it came into existence, or that it did come into existence.

We simply know that it is *there*. We further recognize two general forms of matter — ordinary gross matter and the ether. Gross matter is of various kinds — some 70 odd elements — while the ether is apparently uniform, extending through all space, and in fact occupying all space with the exception of that occupied by the atoms of gross matter. We further recognize that none of this matter is ever at rest, but that it changes its position relatively to space incessantly — or it is always in motion. We recognize therefore in the universe, matter and motion, and these are the only elements of which we have any cognizance.

The ether is the connecting medium of the universe, through and by means of which, gross matter imparts its motion to other particles at a distance. We shall see that matter in motion results in a conception called *force*, and also in a conception called *energy*, and that these three inseparable entities — motion, force, and energy — are transferred to distant points through the medium of the ether. If two particles of matter were separated by an absolute vacuum, i.e., a space containing no matter of any kind, it would be impossible for the motion of one particle to impress itself upon another, or to influence it in any way. It is further evident that, since matter is neither created nor destroyed, and since motion in a body only arises after it has been transferred to it from some other body — the body imparting the motion losing as much as it transfers — the total amount of matter and motion in the universe must always remain constant.

We know very little of the ether except that it is a fluid of extraordinary tenuity and under a very high pressure. What its density and pressure are, we know only roughly, but we know that it is a kind of matter, since it possesses the fundamental property of all matter, viz., inertia. Inertia means literally the helplessness of matter, or the inability of matter, from any virtue within itself, to acquire motion, or when in motion to bring itself to rest.

Some external motion or force is necessary to produce motion or a change of motion in matter. The pressure of the ether, which is a force, must be due to some kind of internal motion in the ether, but of the nature of this motion there is, as yet, hardly a surmise. The atoms of gross matter take up and reflect this motion of the ether and thus become points or centres from which disturbances radiate in all directions. Every atom therefore influences every other atom in the universe, and the observed attractional and repulsional effects are due to these radiations.* The reservoir of motion (energy) is therefore the ether, of which it contains an infinite amount. It has been surmised that the ether is the ground stuff out of which all gross matter has been formed by compression — the density of an element varying with the conditions of pressure, and possibly temperature, under which it was formed. But of such matters we know little or nothing. It has also been suggested that inertia is a property of the ether alone and that gross matter in an ether vacuum would yield to a push without any resistance and cease its motion instantly when the push was removed.

Any motion of a body necessarily sets up a disturbance in the ether and it has been suggested that the reaction against the body of such a disturbance might account for the resistance to motion which we call inertia. But granting such a possibility, we have no explanation of the inertia of the ether. Inertia therefore is at present the one great mystery of nature. It is not enough to say that it is an inherent property of all matter, for there must be some cause. To move a large body — isolated in space — requires a very considerable force and it yields to this force only slowly and reluctantly, but when once in motion there is an equal difficulty in bringing it to rest.

Experiment shows that the resistance to motion of all

*The author has shown with some probability, that attraction and repulsion are due to longitudinal waves in the ether, *v.* "Mechanics of Electricity."

matter is proportional jointly to the mass, or quantity of matter, and to the acceleration, or rate at which the motion changes. Naturally two identical bodies would offer twice the resistance of a single body, and when no forces are applied we should expect the state of motion, or velocity, to remain unaltered both in amount and direction. It is only when the state of motion is being altered that we should expect to meet with any resistance, and such we find to be a fact.

The simplest relation between change of motion and resistance to motion is one of simple proportionality and experiment proves that this is the relation, or the resistance is proportional to the rate at which the motion changes. If we denote the resistance by $r$, this is expressed by the relation, $r = -m\frac{dv}{dt}$. We define a force as the product of a mass by its acceleration. To change the motion of a body, therefore, we shall have to apply a force, $f$, equal and opposite to $r$, and we have the equation $f = m\frac{dv}{dt}$ (1), where $f$ and $\frac{dv}{dt}$ are directed quantities, or vectors, having the same direction. Equa. (1) is Newton's second law of motion which he stated "Change of motion is proportional to the applied force and such change is effected in the direction of the force." It also contains his third law, which is, "The action of the force is equal and opposite to the reaction (of the inertia)." His first law is merely a corollary of the other two.

Equa. (1) may be called the Fundamental Law of the universe, for from it we shall derive all the principles and laws of natural philosophy. There is nothing in it which is axiomatic or *a priori* evident, although Newton's laws have been called "Axiomata Sive Leges Motus."

But the fundamental law, $f = m\frac{dv}{dt}$, is not actually a law, or it does not express *exactly* the relation between the

variables. It is merely the statement of experimental results under certain limited conditions, and even under these conditions the law gives the relations only to a very close approximation. If a force acting upon a body at rest in the air imparts a certain acceleration in unit time, the force will have overcome what we call the inertia of the body and in addition a very slight resistance from the air, provided the force is small. Applying the force continuously we find that the increment of the acceleration is not constant, but is a function of the previously existing velocity. A curve expressing the relation between the force and the acceleration will be nearly a straight line at the beginning, as it would be for any continuous function expressing a relation. But in our present case the curve will fall away because the resistance of the air increases with the velocity. At a certain velocity the resistance is as the square of the velocity, and when the body has a velocity beyond that with which a disturbance travels in the air — about 1100 ft. per second — a vacuum forms behind the body, because the body moves at the same rate as the vacuum tends to close up. A constant force equal to the pressure of the air would therefore give no acceleration, but merely maintain the velocity constant.

Let us now repeat our experiments in an air vacuum. We now find that the curve is very nearly a straight line, or the resistance of the inertia is very nearly proportional to the acceleration. But the body is moving in the ether and any motion in a medium is resisted. The velocity of a disturbance in the ether is about 186,000 miles per second, so that if a body were moving with a greater velocity than this it would require a force equal to the pressure of the ether to maintain this velocity constant without imparting any acceleration. Now the greatest velocity in gross matter which has ever been observed is about 300 miles per second. This is insignificant in comparison with the disturbance velocity of the ether, and we are therefore justified in concluding that our law — verified by experiment —

holds to an extreme degree of approximation for all matter moving with ordinary velocities. But we cannot apply the formula to matter moving with extraordinary velocities. There are certain bodies called electrons, which are possibly ethereal vortical rings of the smallest size possible (without a lumen), which may move with a velocity approaching an ether wave. As such bodies meet with a resistance which increases with the velocity, it has been assumed that their mass increased with the velocity, and it has been lightly stated that the mass, or quantity of matter, in all bodies — gross or ethereal — is not constant, but a function of the velocity! This is an example of latter day metaphysical physics.

The amount of matter in a body is at all times, of course, definite, and can only change by the addition or subtraction of matter.

The energy of a body in motion, called its Kinetic Energy, is defined as half its mass into the square of its velocity, or $T = \frac{mv^2}{2}$. The work done by a constant force is defined as the product of the force into the distance through which it acts in the direction of the force, or $W = fs$. If the force is not constant,

$$W = \int_1^2 f\,ds = \int_1^2 m\frac{dv}{dt}ds = \int_1^2 mv\,dv = \frac{m}{2}\left(v_2^2 - v_1^2\right),$$

or the increase of the kinetic energy is equal to the work done upon the body between any two positions. Kinetic energy and work are thus equivalent, and we may define energy in general as that which is capable of doing work, and the doing of work as the transference of energy from one body to another. All such processes are evidently reversible.

In measuring these quantities certain standard units are selected, usually the gramme for mass, the centimeter for length and the second for time, these units constituting the C. G. S. system. Accordingly the unit of force, called a dyne, is one which acting upon a gramme for a second

imparts to it a velocity of a centimeter a second. The earth's attraction upon a gramme at its surface gives it a velocity of about 981 cms. per second, for every second that it acts, and the force is therefore 981 dynes. The attractional force on a kilogram is 1000 times as much, but the acceleration is the same. Hence all bodies fall (in a vacuum) in the same time. The symbol $g$ is used to express this constant acceleration at the earth's surface.

$$g = 981\frac{cms.}{sec.}. \quad f = m\frac{dv}{dt} = mg = m\ 981\ dynes,$$

where $m$ is expressed in grammes. If the earth and moon were brought to rest and allowed to fall to the sun from the same position, the sun being fixed, they would perform the journey together and reach the sun at the same instant. The accelerations and hence the velocities would be the same at all times.

## 2. Centrifugal Force

Let us suppose a particle of mass, $m$, attached to a string, $OP$, and moving in a circle about $O$ without gravity. In the short time $dt$, it would, if free, move to $A$. Calling $f$ the average pull of the string, both in amount and direction, we have

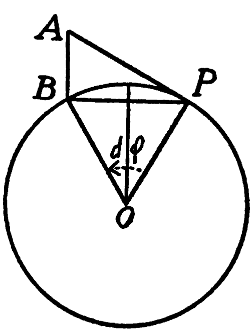

FIG. 1.

$AB = f\frac{dt^2}{2m} = v\,sin\frac{d\varphi}{2}.\,dt$, where $d\varphi$ is the angle $POB$, and $v$ is the velocity at $P$. Developing the sine into its series,

$$f = \frac{2mv}{dt}\left(\frac{d\varphi}{2} - \frac{d\varphi^3}{48} + \text{etc.}\right) = mv\dot{\varphi}\left(1 - \frac{d\varphi^2}{24} + \text{etc.}\right).$$

We shall generally use the Newtonian notation of a super-dot for a velocity and a double super-dot for an acceleration. At the limit where $d\varphi$ becomes zero, $f = m\,v\,\dot{\varphi} = \frac{mv^2}{\rho} = m\rho\dot{\varphi}^2$, where $\rho$ is the radius of curvature of the path at any instant. Hence whenever a body is moving in a curved

path its inertia gives rise to a centrifugal force, or force directed away from the centre of curvature, which is equal to its mass into the square of its velocity, divided by the radius of curvature. This force measures the tension of the string in our example. Whenever a body is moving freely in a curved path, as a planet or a projectile, the component of the force acting upon it which is $\perp$ to the path must always be equal to the centrifugal force and opposite to it. This component obviously cannot influence the velocity of the body but merely deflects its course. A body thrown into the air describes a parabola, and the equation of the curve referred to its vertex is $4ay = x^2$, where $a$ is the distance of the vertex either from the focus or from the directrix. The time of falling from the vertex to the level of the focus is $t = \sqrt{\frac{2a}{g}}$. $x$ at this level is $2a$ and the constant horizontal velocity is $\sqrt{2ag}$. At the vertex the centrifugal force is therefore $\frac{2mag.}{\rho}$ This must be equal to $mg$ and the radius of curvature at the vertex is $2a$.

### 3. Principle of Least Action

We shall now examine the manner in which motions are executed in nature with a view to ascertaining whether any particular economies are effected. When a system is contained in a closed surface across which forces (motions) do not pass, the total energy of the system remains constant and the processes consist only of reversible interchanges between work and kinetic energy within the system. Such a system is said to be conservative since its total energy remains constant. But if during these changes the work is not transformed wholly into gross motion, or gross kinetic energy, but some part of it is expended in producing fine (molecular) vibrations, such as heat (friction), which fine motion is transferred to the ether and thus lost to the system, then the system is non-conservative and its energy does not remain constant.

In a conservative system the kinetic energy of any configuration obviously depends simply upon the position or co-ordinates of that configuration, so that no matter what changes it has undergone, on returning to the same configuration it always has the same kinetic energy. Hence, taking account of friction, the impossibility of perpetual motion in an isolated system.

Let us suppose that a system changes from a certain configuration at the time $t_1$, by its own free motion, to another configuration at the time $t_2$, the coordinates of any one particle of the system at any instant being $x, y, z$. If we vary the motion by guides or constraints so that at any instant a particle instead of occupying its natural position $x, y, z$, occupies a position, $x + \delta x$, $y + \delta y$, $z + \delta z$, where $\delta x$, $\delta y$, $\delta z$ are infinitesimal arbitrary quantities, but the initial and final positions are the same, the difference between the work done in the free paths and in the varied paths is

$$\delta W = \Sigma(X\delta x + Y\delta y + Z\delta z) =$$

$$\Sigma m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right) = \Sigma m\left[\frac{d}{dt}\left(\frac{dx}{dt}\delta x\right) + \right.$$

$$\left.\frac{d}{dt}\left(\frac{dy}{dt}\delta y\right) + \frac{d}{dt}\left(\frac{dz}{dt}\delta z\right) - \left(\frac{dx}{dt}\,d\,\frac{\delta x}{dt} + \frac{dy}{dt}\,d\,\frac{\delta y}{dt} + \frac{dz}{dt}\,d\,\frac{\delta z}{dt}\right)\right],$$

where $X$, $Y$, $Z$ are the components of the force acting upon any particle and $t$ is the independent variable.

The last term is the variation of $T$, the kinetic energy,

for $$\delta T = \delta\Sigma\frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] =$$

$$\Sigma m\left[\frac{dx}{dt}\delta\left(\frac{dx}{dt}\right) + \frac{dy}{dt}\delta\left(\frac{dy}{dt}\right) + \frac{dz}{dt}\delta\left(\frac{dz}{dt}\right)\right].$$

Hence $$\delta\left(W + T\right) = \frac{d}{dt}\Sigma m\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right).$$

Integrating, $$\delta\int_1^2\left(W + T\right)dt = \Sigma m\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right)\Bigg|_{t_1}^{t_2}.$$

Since the variations vanish at the upper and lower limits,

$$\delta \int_1^2 (W + T)dt = 0.$$

Negative work, or stored up energy, is called Potential energy, designated by $V$, and $W = -V$.

Hence
$$\delta \int_1^2 (T - V)dt = 0.$$

This result is known as Hamilton's Principle. It is entirely general and applies to non-conservative as well as conservative systems. Since in a conservative system $\delta W = \delta T$,

$$\delta \int_1^2 (T + W)dt = 2\delta \int_1^2 Tdt = 0.$$

For the natural paths, therefore, these integrals have a stationary value, or for all other infinitely near paths they are either greater or less. As a matter of fact all these integrals are minima, for taking the expression $\delta \int_1^2 Tdt = 0$, it is evident that by causing a particle to execute an infinitesimal loop at any point of the actual path, the integral must be greater than for the actual path, and the actual or free path cannot be a maximum.

The integral $\int_1^2 Tdt$ is called the action of the system, and we have proved that in a conservative system the natural action is less than that in any other infinitely near path. This is known as the Principle of Least Action. It means that the average work, or kinetic energy, for the time, or the time mean of the work or kinetic energy, multiplied by the time, cannot possibly be made less. Or if we desire to effect a given change in any way different from the natural way, we shall have to expend more energy, or do more work than nature does, provided the change is effected in the same time.

This is a remarkable result contained implicitly in the fundamental law.

Let us take the case of a body moving under the in-

fluence of gravity between any two points. Taking the vertex of the path as origin, the time as abscissas and the work as ordinates, the action between 0 and 2 will be represented by the area 023. Since $W = \frac{g^2t^2}{2}$, the action curve 072 will be a parabola, though not of course the parabola of the path. The action, $\int_1^2 W dt = \frac{g^2 t_2^3}{6} = \frac{W_2 t_2}{3}$. The time mean of the work, or the average work for the time is therefore ⅓ of the total work, and the product of this mean by the time, or the area 023 = 0453 = the action, cannot possibly be made less. If the motion is between two points on the same level, corresponding to 1 and 2,

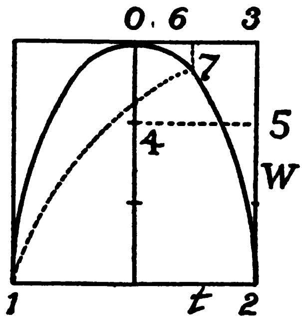


FIG. 2.

the action is zero, for the action between 1 and 0 is a minus area, since $W$ is negative in this part, and the action between 0 and 2 is an equal positive area. There are two free paths by which a body may move between any two points — say points corresponding to 1 and 7 — but the action in each case will be the same, viz., 076 − 023 and −6723. Between points corresponding to 1 and 2 there are two free paths and an infinite number of possible guided paths, but in every case the action is zero. In this case alone, which is an absolute minimum, it is possible for a guided path to have no greater action than the free path, but in all other cases the free action is less than any constrained action, and in all cases the free action is the least possible. Nature, therefore, takes no care of time but is very exacting as to how her energy shall be expended. She insists that the action shall be the least possible. When, however, no energy is expended she becomes economical both as to time and space. $\delta \int_1^2 T dt = \frac{m}{2} \delta \int_1^2 v ds = 0$. When no energy

is expended, $v$ is constant, and $s$ is a minimum as well as $t$, for the path between any two points becomes a straight line. A body under no forces, or a ray of light, moves in a straight line, or they traverse the shortest possible path between any two points in the shortest possible time.

### 4. Brachystochrone and Tautochrone

Whenever energy is expended it is easy to improve upon nature in the matter of time, and we shall construct a path such that a body moving under gravity may pass from one point to another in the least possible time.

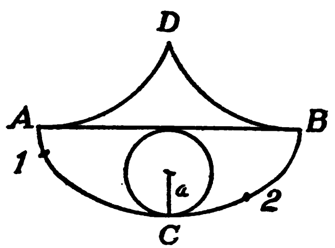


FIG. 3.

Let us suppose (Fig. 3) that our body is at the point 1 and we wish to conduct it to the point 2 in the shortest possible time. It has a velocity $v_0$ at 1, and we take as our base of ordinates a line $AB$ which is $\frac{v_0^2}{2g}$ above 1.

$$t = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{\sqrt{1 + x'^2}}{\sqrt{2gy}} \cdot dy, \text{ where } x' = \frac{dx}{dy}.$$

$$\delta t = \int_1^2 \frac{x'\,\delta dx}{\sqrt{1 + x'^2}\,\sqrt{2gy}} = \left.\frac{\delta x.\, x'}{\sqrt{1 + x'^2}\,\sqrt{2gy}}\right|_1^2 - \int_1^2 \delta x d.\left(\frac{x'}{\sqrt{1 + x'^2}\,\sqrt{2gy}}\right).$$

The variations vanish at the points 1 and 2, so that for $t$ to be a minimum, the last term must vanish, or

$\frac{x'}{\sqrt{1 + x'^2}\,\sqrt{2gy}}$ must be a constant — say $\frac{1}{2\sqrt{ag}}$, where $a$ is a constant to be determined directly. If $\tau$ is the angle which the path at any point makes with the $y$ ordinate, $\sin \tau = \frac{x'}{\sqrt{1 + x'^2}}$. Hence, the required path, in terms of $y$ and $\tau$, is $2a \sin^2\tau = y$. This is the equation of a cycloid

where $a$ is the radius of the generating circle and $y$ is measured from its base line. The solution is entirely general and unique, since a circle can always be found which rolling along a given base line will trace with one of its points a curve which shall pass through two given points, and there is only one such circle. The radius of the generating circle is readily found from the data.

The cyloid has also the property that the time taken in falling from rest to the lowest point, $C$, of the curve, is the same for all starting points. For $dt = \dfrac{ds}{v} = \dfrac{\sqrt{1 + x'^2}}{\sqrt{2g(y - y_1)}}dy,$ where $y_1$ is the ordinate of the starting point.

$$t = \int_{y_1}^{2a} \sqrt{\frac{1 + x'^2}{2g(y - y_1)}}\, dy.$$

But $1+x'^2 = \dfrac{2a}{2a - y}$, and $t = \displaystyle\int_{y_1}^{2a} \sqrt{\frac{a}{g(y - y_1)(2a - y)}} =$

$$2\sqrt{\frac{a}{g}} \sin^{-1} \sqrt{\frac{y - y_1}{2a - y_1}}\Bigg|_{y_1}^{2a} = \pi\sqrt{\frac{a}{g}}.$$

Or the time is independent of the starting point. The time of a complete oscillation is $2\pi\sqrt{\dfrac{4a}{g}}$. Since a cycloid is the involute of two equal cycloids placed above it as in Fig. 3, or since it can be described by fixing a string of length $4a$ at $D$ and wrapping it around the upper cycloids, it is evident that small oscillations of a particle about the lowest point, $C$, will coincide very nearly with those of an ordinary simple pendulum of length $4a$. Hence the time of a small oscillation of a simple pendulum is $2\pi\sqrt{\dfrac{l}{g}}$, $l$ being its length. We can easily derive this tautochronous property of the cycloid in another way. The intrinsic equation of a cycloid is $s = 4a \sin \varphi$, where $s$ is the length of the curve measured from any point and $\varphi$ is the angle between the directions of the curve at the first and any second point. Measuring $s$ from the lowest point, since the acceleration along the

curve is $\ddot{s} = -g \sin \varphi$, $s = -\frac{4a}{g}\ddot{s}$ (1). Such a motion where the force of restitution is proportional to the distance from the position of equilibrium is called *harmonic* motion, and a particle oscillating about the lowest point of a cycloid therefore executes a harmonic motion. Integrating (1),

$$\dot{s} = \sqrt{\frac{g}{4a}(s_1^2 - s^2)}, \text{ where } s_1 \text{ is the starting point.}$$

$$t = \sqrt{\frac{4a}{g}} \sin^{-1} \frac{s}{s_1}\Big|_o^{s_1}, \text{ and the complete period is}$$

$T = 2\pi\sqrt{\frac{4a}{g}}$, which is independent of the starting point.

5.

We shall verify the principle of least action by determining directly what the path of a body moving under gravity must be in order that the action between any two points of the curve shall be a minimum. The body is projected from 1 with a velocity $v_o$ and guided (if necessary) along a curve passing through 2, such that the action between these two points is a minimum. Take as origin of $y$ co-ordinates a base line $OX$, such that $y_1 = \frac{v_o^2}{2g}$.

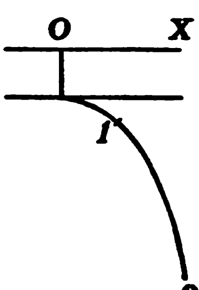

FIG. 4.

Then $$\int_1^2 T dt = \frac{m}{2}\int_1^2 v ds = \frac{m}{2}\int_1^2 \sqrt{2gy}\sqrt{1+x'^2}.dy,$$

and this integral must be a minimum. Considering $y$ the independent variable and varying $x$, we have

$$\delta \int_1^2 \sqrt{2gy}\ \sqrt{1+x'^2}.dy = \int_1^2 \frac{\sqrt{2gy}}{\sqrt{1+x'^2}}\, x'\,\delta dx = 0 =$$

$$\frac{x'\delta x\ \sqrt{2gy}}{\sqrt{1+x'^2}}\Big|_1^2 - \int_1^2 \delta x.d\left(\frac{\sqrt{2gy}}{\sqrt{1+x'^2}}\,x'\right).$$

For the action to be a minimum, $\frac{\sqrt{2gy}}{\sqrt{1+x'^2}}\,x'$ must be a con-

stant, $c$. When $y = \frac{c^2}{2g}$, the curve is horizontal and it does not extend above this level since values become imaginary. Let $c^2 = 2ag$, where $a$ will be determined directly. Hence $x' = \sqrt{\frac{a}{y-a}}$ and $x = 2\sqrt{a}\,\sqrt{y-a} + C$. We have not yet fixed our origin of co-ordinates, but shall take it, so that $C = 0$. Hence the required path is $x^2 = 4a(y - a)$. This is the equation of a vertical parabola with its directrix as the $x$ axis, and $a$ the distance of the vertex from the directrix, or from the focus. The velocity of any projectile at its highest point is thus the same as if it had fallen to that point from the directrix from rest. The required path of least action is therefore a parabola passing through the two points and bearing a certain relation to the initial velocity. Since three conditions fix any conic, the parabola is determined. But this parabola is precisely the free path.

## 6. Central Forces

Let a particle of mass $m$ move subject to a force which radiates from a fixed point. Let $r$ be the distance *from* the point to the particle at any instant and $\dot{\varphi}$ the angular velocity of this radius. The forces acting along a radius are the centrifugal force and the force $f$ radiating from the point. The effective force along a radius is $m\ddot{r}$, and $m\ddot{r} = mr\dot{\varphi}^2 + f$ (1). If $f$ attracts the particle it is negative and opposes the centrifugal force. Integrating (1)

$$\int_1^2 f dr = \overset{2}{\underset{1}{W}} = T_2 - T_1 = \frac{m\dot{r}^2}{2}\Bigg|_1^2 - \frac{mr^2\dot{\varphi}^2}{2}\Bigg|_1^2 + \int_1^2 mr^2\dot{\varphi}d\dot{\varphi}$$

$= m\dot{r}^2 \Big|_1^2 + \frac{mr^2\dot{\varphi}^2}{2}\Big|_1^2 - \int_1^2 \dot{\varphi}\, d\,(mr^2\dot{\varphi})$. The last integral must therefore be zero, and $mr^2\dot{\varphi} = mrv_c$ is constant, where $v_c$ is the circumferential velocity, or the velocity $\perp$ to the radius. It will be noted that the centrifugal force, being an *internal* force arising solely from inertia, can do no work upon the particle, the increase or decrease of

kinetic energy being due solely to the external force, $f$. The product of a directed quantity by its distance from an axis is called the moment of the quantity about that axis. The moment of the angular velocity is evidently the linear circumferential velocity. The moment of a force about an axis is called a Couple. If two equal and opposite forces act at right angles to the extremities of a line which is fixed at its centre, the measure of such a couple is the product of one of the forces into the distance between them, or it is the moment of one of the forces about the extremity of the line joining the two forces. The product of a mass by its velocity is called the momentum of the mass, or it is the quantity of motion in a body.

We have obtained the important principle that a body, under the action of a central force only, preserves the moment of its momentum about the centre constant. The area described by a radius is $\frac{1}{2}\int_1^2 r^2 d\varphi = Ct$, and this area is proportional to the time, which is Kepler's second law. If the body is subject also to circumferential forces, it does not preserve its moment of momentum constant. For the moment of the elementary circumferential force, or the elementary couple, is $m\,\frac{d}{dt}(r^2\dot{\varphi}) = m\,\frac{d}{dt}(rv_c)$, and the time integral of this is $mrv_c\Big|_1^2$.

We have thus a second important principle — the time integral of a couple about a fixed axis is measured by the increase (or decrease) of the moment of momentum about that axis. Or, since the moment of momentum of a body is its moment of inertia about an axis into the angular velocity about the axis, this is evidently the time integral of the couple acting about that axis. Further, since the kinetic energy about any axis is half the moment of inertia about the axis into the square of the angular velocity, this is evidently the angle integral of the couple. For a particle, $m$, the couple at any instant is $mrD_t r\dot{\varphi}$, and the angular

integral is $m \int_1^2 r\dot{\varphi}d(r\dot{\varphi}) = m\frac{r^2\dot{\varphi}^2}{2}\Big|_1^2$, or the angle integral of the couple is measured by the increase of the kinetic energy.

### 7. Atmospheric Circulations

The following problem is fundamental in the theory of the circulation of our atmosphere, or of any planetary circulation. The question is, what path would a moving mass of air (wind) describe on a rotating spheroid, if such mass were practically unhindered by other masses of air, or by friction.

Let us suppose a plastic mass rotating about an axis and subject to its own gravitation. If it were not rotating it would assume a strictly spherical form and the lines of force at the surface would all pass through the centre. The centrifugal force of rotation causes the mass to bulge at its equator and transforms the sphere into a spheroid, the lines of force at the surface being, as before, $\perp$ to the surface. If $\omega$ is the angular velocity of rotation and $\vartheta$ the angle which a radius makes with the equatorial plane, the centrifugal force at its extremity is $r \cos \vartheta\ \omega^2$, and the resolved part of this along the surface, towards the equator, is $r \sin \vartheta \cos \vartheta\ \omega^2$ to a close approximation, if the spheroid differs but little from a sphere. For a particle to be in equilibrium on the surface, the resolved part of the gravitational force along the surface, towards the pole, must be exactly equal to the surface centrifugal component. This gravitational component would evidently restore the spheroid to its original spherical form, immediately the rotation ceased. For a particle to be in equilibrium at any point on the surface it must therefore have exactly the rotational velocity, $\omega$. If the particle has an angular velocity, $\psi$, about the axis, the force urging it along a meridian is $f = r \sin \vartheta \cos \vartheta\ (\omega^2 - \psi^2)$, a positive value denoting acceleration towards the pole and a negative value towards the equator. The moment of momentum about the axis must remain constant since there is no

couple about this axis, or $\cos^2 \vartheta \dot{\psi} = C$. Hence $f = r\ddot{\vartheta} = r \sin \vartheta \cos \vartheta \left(\omega^2 - \frac{C^2}{\cos^4 \vartheta}\right)$. Integrating and putting $\vartheta = \vartheta_0$ when $\dot{\vartheta}$ becomes zero, we have

$$\frac{\dot{\vartheta}^2}{2} = \frac{\cos 2\vartheta_0 - \cos 2\vartheta}{2} \cdot \omega^2 + C^2 (\tan^2 \vartheta_0 - \tan^2 \vartheta).$$

$$\cos 2\vartheta_0 - \cos 2\vartheta = 2\,C\left(\frac{1}{\dot{\psi}_0} - \frac{1}{\dot{\psi}}\right)$$

and $$\tan^2 \vartheta_0 - \tan^2 \vartheta = \frac{1}{C}\left(\dot{\psi}_0 - \dot{\psi}\right).$$

Hence $\dot{\vartheta}^2 = C\,(\dot{\psi}_0 - \dot{\psi})\left(1 - \frac{\omega^2}{\dot{\psi}_0 \dot{\psi}}\right)$ (1). There are two values of $\dot{\psi}$ which makes $\dot{\vartheta}$ vanish. One of these we have already taken as $\dot{\psi}_0$, and the other is $\frac{\omega^2}{\dot{\psi}_0}$. The path is tangent to a parallel of latitude at these points and the whole curve must lie between these two parallels. Writing the angular velocities at the upper and lower parallels as $\dot{\psi}_u$ and $\dot{\psi}_o$, we have $\dot{\vartheta}^2 = \frac{C}{\dot{\psi}}\,(\dot{\psi}_o - \dot{\psi})\,(\dot{\psi} - \dot{\psi}_u)$ (2). We shall call linear velocities along a parallel, horizontal velocities, and linear velocities along a meridian, polar velocities, designated by $v_h$ and $v_p$. Since $\dot{\psi} = \frac{v_h}{r \cos \vartheta} = \frac{C}{\cos^2 \vartheta}$, we have from (2), $v_p^2 = \frac{1}{v_h^2}\,(v_h^2 - v_0^2)\,(v_u^2 - v_h^2)$ (3).

It must be borne in mind that these are absolute horizontal velocities—not relative to the earth. Designating the relative horizontal velocity by $v_{rh}$, $v_{rh} = r \cos \vartheta\,(\omega - \dot{\psi}) =$

$$r \cos \vartheta\, \omega - \frac{-rC}{\cos \vartheta} = \frac{r^2\,\omega C}{v_h} - v_h \text{ (4).}$$

Let $v_r$ be the total velocity relative to the earth. Then, from (3) and (4),

$$v_r^2 = \frac{r^4\,\omega^2\,C^2}{v_h^2} - 2\,r^2\,\omega\,C + v_h^2 + v_u^2 - v_h^2 - \frac{v_0^2\,v_u^2}{v_h^2} + v_0^2.$$

Now $r^4\,\omega^2\,C^2 = v_0^2\,v_u^2$, since $C = \frac{v_0 \cos \vartheta_0}{r} = \frac{v_u \cos \vartheta_u}{r}$, and

$\omega^2 = \psi_0 \psi_u$. Hence $v_r^2 = v_0^2 - 2r^2\omega C + v_u^2$ (5). Or the velocity relative to the earth is constant. It will be seen that the maximum polar velocity occurs on that parallel where there is no poleward acceleration, or where $\psi = \omega = \sqrt{\psi_0 \psi_u}$. This maximum polar velocity is $v_u - v_0$. Designating the absolute horizontal velocity at this parallel of equilibrium by $v_e$, $v_e^2 = v_u v_0$, or the horizontal velocity at this point is the geometric mean of the extreme horizontal velocities. Since $\cos^2 \vartheta_0\psi_0 = \cos^2 \vartheta_u\psi_u$ and $\psi_u = \frac{\omega^2}{\psi_0}$, $r \cos \vartheta_0\psi_0 = r \cos \vartheta_u\omega$, or the absolute horizontal velocity at the lower limit is equal to the velocity of the earth at the upper limit, and the absolute horizontal velocity at the upper limit is equal to the velocity of the earth at the lower limit. The motion is thus completely determined.

As on all planets the temperature is greater on the whole at the equator and becomes gradually less towards the poles, the atmosphere rises at the equator and is replaced by other portions flowing along the surface from north and south. Such streams, effected by differences of temperature, will be forced to execute such paths as we have just determined, and as in crowds when a general trend is once established there is little mutual interference, so the circulation of the air must approximate closely to the dynamical factors.

The equatorial circulation of a planet is shown in Fig. 5. The currents flowing towards the equator are at first close to the surface but are continually deflected — to the right in the northern hemisphere, to the left in the southern hemisphere. They do not reach the equator, but are deflected due west at a high level. At the parallels of equilibrium, indicated by the dotted lines, the directions are due north and south, the upper currents going poleward while the surface currents are towards the equator. The limiting parallels, north and south, are functions of the difference of temperature between these parallels and also

of the rotational velocity of the planet. Since the curvature of the paths is a minimum nearest to the equator, the general trend of this circulation is constantly to the west.

There is likewise a flat polar circulation the extent and characteristics of which are determined to a certain extent by the elements we have just discussed. Between the equatorial and polar circulations is the temperate circulation composed of several partly independent and not sharply

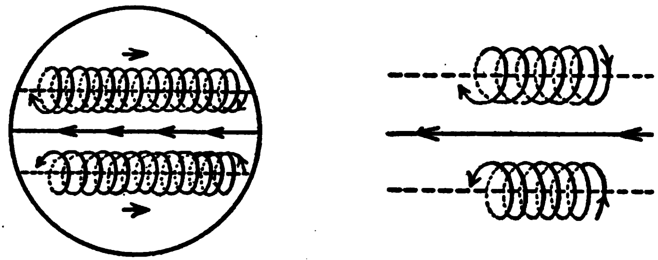

FIG. 5.

differentiated zones. The temperate circulation as a whole moves towards the east with varying northerly and southerly components. As Lord Kelvin has pointed out, there is on the whole a slow shifting, due to friction, of the surface currents towards the poles with a counterbalancing slow shifting at higher levels from the border of the polar circulation to that of the equatorial circulation. The circulation of any planetary atmosphere is thus differentiated into six distinct circulations, the borders of which are very sharply marked. The currents of the equatorial circulation are very constant both as to their intensities and the shapes of their paths, the polar circulation less so, while the temperate circulation is still less stable. It is hardly necessary to state that what would be a constant and stable condition in all the circulations, provided our postulated conditions existed, viz., that the earth had a homogeneous surface and were symmetrically heated about the equator, does not actually exist because

the inclination of the sun to the equator is constantly shifting and because the surface is irregularly divided into land and water, the land being of varying altitudes. In the ideal conditions the equatorial circulations would never reach the equator, while in the actual conditions they frequently cross it. This leads at times to cyclones and various other abnormal disturbances, a fuller discussion of which will be found in "The Atmosphere," by the author. It is interesting to note that Dr. Percival Lowell has observed "Faint lacings. . . criss-crossed by darker lines" in the equatorial zones of both Jupiter and Saturn. It is quite possible that these are cloud streams in their equatorial circulations, and a glance at Fig. 5 shows that they might have just such an appearance when viewed in a telescope.*

From the law of constant moments of momentum, $\ddot{\psi} = 2C \sec^2 \vartheta \tan \vartheta \dot{\vartheta} = 2\dot{\psi} \tan \vartheta \dot{\vartheta}$, and $R \cos \vartheta \ddot{\psi} = 2R \dot{\psi} \sin \vartheta \dot{\vartheta}$. Putting $\dot{\psi}_r$ for the relative angular horizontal velocity, or $\dot{\psi}_r = \dot{\psi} - \omega$, we have $R \cos \vartheta \ddot{\psi} = 2R (\dot{\psi}_r + \omega) \sin \vartheta \dot{\vartheta}$, and $R\ddot{\vartheta} = -R \sin \vartheta \cos \vartheta . 2\omega \dot{\psi}_r$ approximately, if $\dot{\psi}_r$ is small compared with $\omega$. Hence we may write approximately, $R \cos \vartheta \ddot{\psi} = 2R\omega \sin \vartheta \dot{\vartheta}$. Consequently if $\rho$ be the radius of curvature of the path and $v_r$ the relative velocity at any point, $\frac{v_r^2}{\rho} = 2\omega \sin \vartheta v_r$, or $\rho = \frac{v_r}{2\omega \sin \vartheta}$. The curvature of the path is therefore nearly proportional to the sine of the latitude and inversely proportional to the relative velocity. This result was first given by Ferrel.

### 8. Motion of Rigid Masses

We have hitherto considered the motion of particles, or of masses of matter supposed concentrated into a mathematical point. We shall now investigate the motion of masses having definite dimensions. We can consider a rigid body as made up of an infinite number of particles which are held together by an unyielding non-material frame.

*Dr. Percival Lowell. Popular Astronomy. April, 1910.

In any field of force every particle will be subjected to a force, $F$, which we shall call the applied force. It cannot obey this force, as a detached particle would do, by reason of its fixed connections, but the *effective* force on each particle will be the geometrical resultant of the applied force and the sum of the reactions of the neighboring particles, and the particle will obey this effective force the same as if it were free. That is, the actual infinitesimal path, $ds$, will be in the direction of this force, and it will oppose to this force its inertia, measured by $m\frac{d^2s}{dt^2}$, which likewise measures the effective force. Summing all the forces we have three groups — the applied forces, the reactions of the neighboring particles, and the forces of inertia or the effective forces. Now the sum of the reactions among the particles must be zero, since there is no relative motion between them. Hence the geometric sum of all the applied forces must be equal to the geometric sum of all the effective forces. Or $\Sigma F = \Sigma m\frac{d^2s}{dt^2}$, where $ds$ is the actual elementary path of each particle, and $\Sigma$ signifies geometric sum — not algebraic sum. This is D'Alembert's Principle.

Taking any $\perp$ axes, and resolving each applied force, $F$, into $X$, $Y$, $Z$, parallel to these axes, we have

$$\Sigma X = \frac{d^2}{dt^2}\left(\Sigma mx\right), \Sigma Y = \frac{d^2}{dt^2}\left(\Sigma my\right), \Sigma Z = \frac{d^2}{dt^2}\left(\Sigma mz\right).$$

Putting $\Sigma mx = M\bar{x}$, $\Sigma my = M\bar{y}$, $\Sigma mz = M\bar{z}$, where $M$ is the total mass, these equations determine a point in the body having co-ordinates $\bar{x}, \bar{y}, \bar{z}$, and this point is called the *centre of inertia*, or centre of mass. It is evidently a fixed point in the body, irrespective of whatsoever forces act upon it.

The interpretation of these equations is that if we transfer every applied force to the centre of inertia, parallel to itself, the geometrical sum of these forces will be equal to a single force acting upon the entire mass considered as concentrated at this point and this force will, of course,

be equal and opposite to the inertianal force of such a concentrated mass. In other words, the motion of the centre of inertia will be same as the motion of a material point or particle of mass $M$, under a force which is the geometrical resultant of all the applied forces acting at this point parallel to their original directions.

We have next to consider that the applied forces do not act at the centre of inertia, but on the several particles. Since $\Sigma Y = \Sigma m \frac{d^2y}{dt^2}$, $\Sigma xY = \Sigma mx \frac{d^2y}{dt^2}$ and $\Sigma yX = \Sigma my \frac{d^2x}{dt^2}$, $xY$ is the moment of the force $Y$ about the axis of $Z$, and $yX$ is the moment of $X$ about this axis. Hence $\Sigma(xY - yX) = \Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)$ means that the couple about the $z$ axis due to all the applied forces is equal and opposite to the couple about this axis due to the inertianal forces. We have then,

**A.**

$$\Sigma X = \Sigma m \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}\left(\Sigma mx\right) = \frac{d^2}{dt^2} M\underline{x}$$

$$\Sigma Y = \Sigma m \frac{d^2y}{dt^2} = \frac{d^2}{dt^2}\left(\Sigma my\right) = \frac{d^2}{dt^2} M\underline{y}$$

$$\Sigma Z = \Sigma m \frac{d^2z}{dt^2} = \frac{d^2}{dt^2}\left(\Sigma mz\right) = \frac{d^2}{dt^2} M\underline{z}.$$

and

**B.**

$$\Sigma\left(yZ - zY\right) = \Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = \frac{d}{dt}\Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)$$

$$\Sigma\left(zX - xZ\right) = \Sigma m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) = \frac{d}{dt}\Sigma m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right)$$

$$\Sigma\left(xY - yX\right) = \Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = \frac{d}{dt}\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right).$$

Equas. A state that the sum of the momenta of all the particles in any direction is equal to the component in that direction of the momentum of the total mass moving with the velocity of the centre of inertia.

Equas. B state that the derivative with respect to the time of the moment of momentum about any axis is equal to the couple about that axis, a result which we have already obtained. It follows that the motion of a rigid body under the action of any forces can always be resolved into a translational motion of the centre of inertia and a rotation about an axis through that centre. It is further evident that these two motions are entirely independent of each other, so that if we oppose the translational motion, the rotation will occur as before, and if we prevent the rotation the translational motion will be uninfluenced.

We can arrive at these results more simply as follows: The applied forces can be reduced (geometrically) to a single force acting through some line within or without the body (but if without the body to be considered as rigidly connected with it), and the geometric sum of all the inertianal forces is a single force acting through this same line, but in the opposite direction. Dropping a $\perp$ from the centre of inertia to this line, and applying to the centre of inertia a force equal and parallel to the resultant of the applied forces and two forces equal to half this force, but opposite in direction, to the extremity of the $\perp$ and an equal distance on the other side of the centre of inertia respectively, this system will be in equilibrium. But this system combined with the resultant of the applied forces is equivalent to a single force acting at the centre of inertia equal and parallel to the resultant, together with a couple about an axis through the centre of inertia. Likewise, reversing all the directions, the resultant of all the inertianal forces is equivalent to an equal and parallel force acting at the centre of inertia, together with a couple about an axis through this centre. The axis of the couple is $\perp$ to the plane

through the line of action and centre of inertia, and the intensity of the couple is the moment of the resultant of all the applied forces about the centre of inertia.

If the resultant of the applied forces passes through the centre of inertia there can be no rotation. A homogeneous sphere in a centrally attracting field can acquire no rotation and is said to be centrobaric. That is, the resultant line of attraction of an attracting point always passes through the centre of inertia of the sphere. This is evident from symmetry. Likewise no body, whatsoever its shape, can acquire a rotation in a uniform parallel field, such as the field at the earth's surface.

It is to be observed that generally the motion of the centre of inertia is not the same as if the whole mass were first concentrated into its centre of inertia and *then* acted upon by the field. What we have proved is that for any field it is the same as if the applied forces were applied to the total mass at the centre of inertia, parallel to their original directions. However, in certain fields the result will be the same in either case. In uniform parallel fields such will obviously be the case, and also (*v.* Art. 24) when the forces tend to a fixed centre and vary as the distance from that centre.

If we define the centre of gravity as a point in a body such that when fixed the body is not rotated by the field in any position, it is evident that when there is such a point it is the centre of inertia, but that generally there is no such point. The earth being a spheroid is not centrobaric for central fields and therefore has no such point. The sun's field and the moon's field both produce rotations of the earth which result in the precession of the equinoxes. Usually the term Centre of Gravity is taken as synonymous with Centre of Inertia. Since every mathematical conception should have a single name and as there are other fields than gravitational, it would seem advisable to employ the term Centre of Inertia alone.

Using polar co-ordinates, $x = r \cos \vartheta$, $y = r \sin \vartheta$, where

$\vartheta$ is the angle between a radius in the $x$, $y$ plane and the $x$ axis, $\frac{d^2x}{dt^2} = -r\left(\sin\vartheta\ddot{\vartheta} + \cos\vartheta\dot{\vartheta}^2\right)$, $\frac{d^2y}{dt^2} = r\left(\cos\vartheta\ddot{\vartheta} - \sin\vartheta\dot{\vartheta}^2\right)$. Hence the couple about the $z$ axis is $\Sigma mr^2\ddot{\vartheta} = \ddot{\vartheta}\Sigma mr^2$. The integral $\Sigma mr^2$ is called the moment of inertia of a body about an axis $\perp$ to $r$. Letting $\Sigma mr^2 = M\bar{r}^2$, $\bar{r}$ is called the Radius of Gyration, and it is the average radius which would give the same moment of inertia if the whole mass were concentrated at its extremity. A couple is therefore measured by the moment of inertia into the angular acceleration about an axis.

## 9. Moments of Inertia

Taking some point in a body as origin of rectangular co-ordinates, let us draw radii in all directions from the origin of such lengths that the moment of inertia about any radius as an axis shall be equal to the square of the reciprocal of the radius, or $I = \frac{1}{r^2}$, where $I$ is the moment of inertia. The locus of the extremities of these radii will be a surface. Designating the moments of inertia about the axes as $I_x$, $I_y$, $I_z$, $I_x = \Sigma m\,(y^2 + z^2) = \frac{1}{r_1^2}$, $I_y = \Sigma m\,(x^2 + z^2) = \frac{1}{r_2^2}$, $I_z = \Sigma m\,(x^2 + y^2) = \frac{1}{r_3^2}$. The sum of these moments is $2\,\Sigma mr^2$, a constant, where $r$ is the distance of any element from the origin, and $r_1$, $r_2$, $r_3$ refer to the momental surface. We have taken any axes, so that our surface has the property that the sum of the squares of the reciprocals of any three $\perp$ radii is constant. Such a property belongs to an ellipsoid alone. For let $\alpha_1, \beta_1, \gamma_1, : \alpha_2, \beta_2, \gamma_2; \alpha_3, \beta_3, \gamma_3$ be the direction angles of any three mutually $\perp$ radii referred to the principal axes of an ellipsoid, $a$, $b$, $c$. Then

$$\frac{1}{r_1^2} = \frac{\cos^2\alpha_1}{a^2} + \frac{\cos^2\beta_1}{b^2} + \frac{\cos^2\gamma_1}{c^2} \text{ etc.,}$$

and $$\Sigma\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2}\right) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Hence, at any point of any body it is possible to construct an ellipsoid with this point as a centre such that the square of the reciprocal of any radius is equal to the moment of inertia of the body about that radius as an axis. The ellipsoid corresponding to any point is called the Momental Ellipsoid for that point. The principal axes of this ellipsoid are called the principal axes of inertia of the body for that point. The principal axes of inertia corresponding to the centre of inertia are called simply the Principal Axes of the body, and the moments about these axes are the Principal Moments of Inertia. Generally the three principal moments of inertia have different values and such bodies are said to be triaxial. When two of the moments are equal, the body is biaxial and when all three are equal the body is uniaxial.

Taking any axis about which we wish to find the moment of inertia as the $z$ axis and $x$, $y$ as the co-ordinates of the centre of inertia and $x^1$, $y^1$ as the co-ordinates of an element referred to parallel axes through the centre of inertia, since $x = \underline{x} + x^1$ and $y = \underline{y} + y^1$, the moment of inertia about our axis is $\Sigma m\ (x^2 + y^2) = \Sigma m\ (x^2 + y^2) + \Sigma m\ (x'^2 + y'^2)$ since $\Sigma m x^1 = \Sigma m y^1 = 0$. Hence the moment of inertia about any axis is equal to the moment about a parallel axis through the centre of inertia plus the moment of the whole mass concentrated into the centre of inertia about our axis. The moment of inertia about an axis through the centre of inertia is therefore less than that about any other parallel axis.

## 10. Impulsive Forces

We have already seen that a force acting continuously is measured by the mass it acts upon into the acceleration it produces in the mass in unit time. We now wish to determine how a force acting only for a brief interval may be measured. A force cannot of course act instantaneously or

for absolutely no time, for in such a case to produce any finite effect the force would have to be infinite and there is no such thing as an infinite force.

Two elastic balls of mass, $m_1$ and $m_2$, and velocities $v_1$ and $v_2$, meet, going either in the same or opposite directions. We shall suppose that no energy is lost by the impact or no heat developed. The velocity of the first ball is changed, not instantaneously, but in an exceedingly short interval of time, from $v_1$ to $v_1'$, and that of the second ball from $v_2$ to $v_2'$. During the short interval that they are in contact they must move with the same velocity, $v$, and this velocity is the average velocity while the change in the velocities is being effected. During the time of contact the first ball has changed its velocity from $v_1$ to $v_1'$ and its average velocity during this time must have been $\frac{v_1 + v_1'}{2}$, while the average velocity of the second ball was $\frac{v_2 + v_2'}{2}$. The kinetic energy of the system remains unchanged, so that $\frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1 v_1'^2}{2} + \frac{m_2 v_2'^2}{2}$.

Hence $m_1\left(\frac{v_1 + v_1'}{2}\right)(v_1 - v_1') = m_2\left(\frac{v_2 + v_2'}{2}\right)(v_2' - v_2)$.

But $\frac{v_1 + v_1'}{2} = \frac{v_2 + v_2'}{2} = v$, and $m_1(v_1 - v_1') = m_2(v_2' - v_2)$.

We have thus the measure of a force which acts for a short time and which is called an impulsive force or an impact. The measure is the increase (or decrease) of the momentum which it produces in a body.

## 11. Pendulum

We have seen that the time of a complete small oscillation of a suspended particle is $2\pi\sqrt{\frac{l}{g}}$. If we have a rigid body of mass $M$ oscillating about a horizontal axis, the gravitational couple is $Mgh \sin \vartheta$, where $h$ is the distance

of the centre of inertia from the axis and $\vartheta$ the angle it makes with the vertical. If $I$ is the moment of inertia about the axis, $I\ddot{\vartheta} = -Mgh \sin \vartheta$ (1). If $k$ is the radius of gyration about a parallel axis through the centre of inertia, $I = M(k^2 + h^2)$. For a small oscillation $\sin \vartheta$ is sensibly equal to $\vartheta$. Integrating (1), $\dot{\vartheta} = -\sqrt{\dfrac{gh}{k^2 + h^2}}\ \sqrt{\vartheta_1^2 - \vartheta^2}$, where $\vartheta_1$ is the maximum excursion. Integrating again, $t = \sqrt{\dfrac{k^2 + h^2}{gh}}\left(\dfrac{\pi}{2} - \sin^{-1}\dfrac{\vartheta}{\vartheta_1}\right)$. The time of a complete oscillation is $T = 2\pi\sqrt{\dfrac{k^2 + h^2}{gh}}$ and the length of the equivalent simple pendulum is $l = \dfrac{k^2 + h^2}{h}$.

If we wish to find the time for any amplitude we may proceed as follows: Integrating (1) $\dot{\vartheta} =$

$$\sqrt{\frac{g}{l}}\ \sqrt{2(\cos\vartheta - \cos\vartheta_o)} = 2\sqrt{\frac{g}{l}}\sqrt{\sin^2\frac{\vartheta_o}{2} - \sin^2\frac{\vartheta}{2}}.$$

Let $\sin\dfrac{\vartheta_o}{2}\sin\varphi = \sin\dfrac{\vartheta}{2}$, where $\varphi$ is an auxiliary angle. When $\vartheta = 0$, $\varphi = 0$ and when $\vartheta = \vartheta_o$, $\varphi = \dfrac{\pi}{2}$. Hence $dt =$

$$\sqrt{\frac{l}{g}}\ \frac{d\varphi}{\sqrt{l - \sin^2\frac{\vartheta_o}{2}\sin^2\varphi}}.$$

By the binomial theorem the radical can be developed into

$$1 + \frac{1}{2}\sin^2\frac{\vartheta_o}{2}\sin^2\varphi + \frac{1}{2}\cdot\frac{3}{4}\sin^4\frac{\vartheta_o}{2}\sin^4\varphi + \text{etc.}$$

$$\text{Now} \int_0^{\frac{\pi}{2}} \sin^{2n}\varphi\, d\varphi = \frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ \ldots\ \cdot\frac{2n-1}{2n}\right).$$

Hence the time of a complete swing is

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2\sin^2\frac{\vartheta_o}{2} + \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\sin^4\frac{\vartheta_o}{2} + \text{etc.}\right]$$

From this rapidly converging series we can determine the time for any amplitude as closely as we please.

## 12. Centres of Oscillation and Percussion

An impulsive force acting upon a body obviously imparts to it an impulsive velocity of the centre of inertia, or an impulsive momentum of the entire mass considered as concentrated at this centre, together with an impulsive moment of momentum about an axis through this centre.

If a body (Fig. 6) is struck a blow at $B$ in the direction $AB$, the translational motion of the centre of inertia, $G$, is the same as if the entire mass were concentrated there and the force acted on it parallel to its direction, and the rotational moment of momentum about an axis through $G$ is the same as if this centre were fixed. The impulsive velocity, $v$, of the centre of inertia, parallel to $AB$, will move it a distance $vdt$ in the first small interval of time, and the rotation about an axis through $G$, $\perp$ to the plane $ABG$, will turn the body through an angle $d\vartheta$ in this time. This is

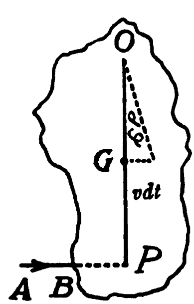


FIG. 6.

equivalent to a rotation about a parallel axis through some point $O$ in the line $OG$, $\perp$ to $AB$. Calling $OP$, $l$, and $OG$, $h$, the moment of the blow $F$ about $G$ is measured by the impulsive moment of momentum about the axis through $G$, or $F\,(l - h) = Mk^2\,\dot{\vartheta}$ and $\dot{\vartheta} = \dfrac{F(l - h)}{Mk^2}$, where $k$ is the radius of gyration about the axis through $G$. The blow $F$ is also measured by the impulsive momentum of the centre of inertia, or $F = Mv = Mh\dot{\vartheta} = Mh\,\dfrac{F(l - h)}{Mk^2}$, and $l = \dfrac{k^2 + h^2}{h}$. The axis of spontaneous rotation for the first instant, or the instantaneous axis, is therefore at a distance from the line of the blow equal to the length of the equivalent simple pendulum. Relative to an axis through $O$, the point $P$ is the centre of oscillation, and it is also a

centre of percussion, for a blow applied at this distance from the axis of spontaneous rotation would cause no reaction upon it. If we strike a ball with a bat or a tree with an axe at a distance from our hands equal to the length of the equivalent simple pendulum, we experience no "sting"; otherwise we do. The subsequent motion of the body consists simply of the initial impulsive velocity of the centre of inertia, and the impulsive rotation about the axis through it, with the superposed effects of subsequent forces. The trajectory of a struck ball is determined by the initial velocity of the centre of inertia, the spin not being influenced by gravity.

If $GP = h^1$, $l = h + h^1$ and $hh^1 = k^2$. Hence the points $O$ and $P$ are mutually centres of oscillation and percussion, one for the other, and the times of oscillation (under gravity) are the same for either suspension. This gives a means of determining the length of the equivalent simple pendulum.

### 13. Harmonic Motion

Distant bodies influence each other by gravitational (longitudinal) waves through the ether. Since such forces radiate equally in all directions from a central point, the intensity of the force must fall off as the square of the distance, or the simple geometrical law must hold. Since two equal bodies radiate twice the energy of a single body, the force must be proportional to the mass, for the same distance. Again the action of such a force upon two equal distant bodies must be twice the action upon a single body. Hence we have almost axiomatically Newton's law of attraction — "All bodies attract each other with a force proportional to the product of their masses and inversely as the square of the distance, or $f = \frac{MM^1}{r^2}$." The action is purely mutual, and the forces urging each body are precisely equal, and opposite.

Let us consider the mutual force between a homogeneous

spherical shell and unit mass at any point. Let $\sigma$ be the density of the matter on the shell, or mass per unit surface, and $h$ the distance of the point from the centre of the shell, $r$ its radius, $\rho$ the distance of the point from any element of the shell and $\vartheta$ the angle between any radius and $h$. From symmetry it is evident that the resultant attraction of the shell must be along $h$. An elemental ring, concentric with the axis $h$, has a mass $2\pi\sigma r^2 \sin \vartheta d\vartheta$. The component along $h$ of every such elemental ring is

$$\frac{2\pi\sigma r^2 \sin \vartheta}{\rho^3} (h - r \cos \vartheta)\, d\vartheta.$$

Since $\rho^2 = h^2 + r^2 - 2rh \cos \vartheta$, $\sin \vartheta \, d\vartheta = \dfrac{\rho d\rho}{rh}$

and $h - r \cos \vartheta = \dfrac{h^2 - r^2 + \rho^2}{2h}$.

The total attraction is

$$\frac{\pi\sigma r}{h^2} \int_{h-r}^{h+r} \frac{(h + r)\,(h - r)}{\rho^2}\, d\rho + \frac{\pi\sigma r}{h^2} \int_{h-r}^{h+r} d\rho \quad (1).$$

There are three cases: If $h > r$, we have $f = \dfrac{4\pi\sigma r^2}{h^2}$: if $h = r$, $f = 4\pi\sigma$: and if $h < r$, we have to take the limits as $r + h$ and $r - h$, and the integral becomes

$$-\frac{2\pi\sigma r}{h} + \frac{2\pi\sigma r}{h} = 0.$$

Hence any homogeneous spherical shell attracts, and is attracted by, any external mass precisely as if its mass were concentrated at its centre, but within the shell it exercises no attraction, or there is no force. And any sphere made up of homogeneous spherical shells attracts in the same way.

Let us suppose a homogeneous sphere with two diameters bored out $\perp$ to each other. The attraction of the sphere on unit mass at the surface is $\dfrac{M}{R^2} = \dfrac{4\pi R\rho}{3}$, and the attraction at any level in the interior is $\dfrac{4\pi r\rho}{3}$, where $r$ is the distance from the centre. Hence if we drop unit mass

into one of these holes, it will oscillate harmonically between two diametrically opposite points of the surface, the motion being harmonic because the force is proportional to the distance from the centre. Let $2\sqrt{\frac{\pi\rho}{3}}$ be $k$. If we project the mass along the surface with a velocity, $v = kR$, which makes the centrifugal force just equal to the attraction, it will revolve about the sphere, just grazing the surface. The time for the outside mass to traverse a quadrant is $\frac{\pi}{2k}$, and the time to reach the centre is the same.

$$\ddot{r} = -k^2r,\ \dot{r} = k\sqrt{R^2 - r^2} \text{ and } t = \frac{1}{k}\sin^{-1}\frac{r}{R}\Big|_o^R.$$

Hence the outside and inside masses will regularly meet as the outside mass passes over each hole. Using the holes as axes of $x$ and $y$, the positions of the inside masses will be the co-ordinates of the outside mass.

The inside bodies execute simple harmonic motions, while the outside body executes a compound harmonic motion made up of two equal simple harmonic motions $\perp$ to each other. Since $t = \frac{1}{k}\sin^{-1}\frac{r}{r_o}\Big|_o^{r_o}$, where $r_o$ is some level within the sphere from which we drop a body, the body thus dropped will have the same period as if dropped from the surface. Thus the period of any harmonic motion is independent of the amplitude of the motion. It is for this reason that such a motion is called harmonic. The vibrating parts of all musical instruments execute harmonic motions. Otherwise the period (pitch) would change with the intensity (amplitude) and music or harmony would become impossible.

Two $\perp$ simple harmonic motions with the same period and amplitude, and one of the motions a quadrant in advance (or behind) of the other, result in a circular harmonic motion. If the amplitudes are not equal we have elliptic harmonic motion. Since $t = \frac{\pi}{2k} + \frac{1}{k}\sin^{-1}\frac{x}{R}$ for

one mass when $t = \frac{1}{k}\sin^{-1}\frac{y}{r_0}$ for the other, we may write

$$x = R \sin\left(kt - \frac{\pi}{2}\right) = R \cos kt,$$

$y = r_0 \sin kt$ and $\frac{x^2}{R^2} + \frac{y^2}{r_0^2} = 1$, or the path for the compound motion is an ellipse. The angle in these expressions is called the phase of the motion and the difference between the phases of the two components is always $\frac{\pi}{2}$. The reciprocal of the period is called the frequency, or the number of vibrations in a second. The complete period is $\frac{2\pi}{k}$. It is evident that any number of simple harmonic motions making any angles with each other, but all having a common centre, can be compounded. If there is a common period the result will be a steady elliptic motion. If the periods are different the resultant motion will be continually shifting, forming what are known as Lissajon's curves. If the periods have a common multiple the changes will periodically repeat themselves.

## 14. Tidal Forces

A homogeneous ring revolves about an attractional centre, $S$, which is in its plane, and the plane of the ring is $\perp$ to its orbit. The distance $SC = D$ is constant and $\varphi$ is the angle any element of the ring makes with $CA$. $D$ is so great that all lines from $S$ to the ring may be regarded as sensibly parallel. The centrifugal force for any element is $(D + r\cos\varphi)\ \psi^2 r d\varphi$, where $\psi$ is the orbital angular velocity. Integrating, we find that the

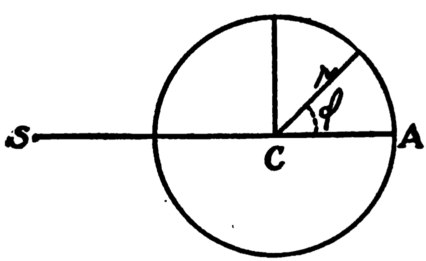


FIG. 7.

centrifugal force for the outer half is $\pi r D\psi^2 + 2r^2\psi^2$, and for the inner half $\pi r D\psi^2 - 2r^2\psi^2$. Calling the mass of the ring, $M$, these two centrifugal forces are

$$\frac{M}{2} D\psi^2 + \frac{M}{2} \cdot \frac{2r}{\pi} \psi^2 \text{ and } \frac{M}{2} D\psi^2 - \frac{M}{2} \cdot \frac{2r}{\pi} \psi^2.$$

$\frac{2r}{\pi}$ is the distance of the centre of inertia of a half ring from $C$, so that the rotational centrifugal force for each half of the ring is the same as if its mass were concentrated at its centre of inertia.

Let $f$ be the acceleration due to the central attraction at unit distance. Then the total attraction of the ring is $\int \frac{frd\varphi}{(D + r \cos \varphi)^2} = \frac{r^2 f \sin \varphi}{(D^2 - r^2)(D + r \cos \varphi)} - \frac{Drf}{(D^2 - r^2)^{\frac{3}{2}}} \cos^{-1} \frac{r + D \cos \varphi}{D + r \cos \varphi}$. Taking the limits for the outer half, and considering that $(D^2 - r^2)$ is nearly equal to $D^2$ and the angle $\cos^{-1} \frac{r}{D}$ is nearly equal to $\frac{\pi}{2}$, we have for the attraction on the outer half $-\frac{\pi r f}{D^2} + \frac{2fr^2}{D^3}$. Likewise the attraction on the inner half is $-\frac{\pi r f}{D^2} - \frac{2fr^2}{D^3}$. If the total centrifugal force, which is $MD\psi^2$, balances the total attractional force, which is $-\frac{Mf}{D^2}$, all the forces will be in equilibrium, and the motion will be equivalent to a mass concentrated at $C$ revolving about $S$ at a constant distance with constant angular velocity. But we have a force acting on the outer half, $\frac{Mr}{\pi}\left(\frac{f}{D^3} + \psi^2\right)$, which is balanced by an equal and opposite force acting on the inner half, $-\frac{Mr}{\pi}\left(\frac{f}{D^3} + \psi^2\right)$. These two forces, therefore, tend to pull the ring apart in either direction from the centre, the inner half being urged towards the attracting body, while

the outer half is urged directly away from it. If the ring is not perfectly rigid, and perfectly rigid bodies do not exist, it will be deformed into an oval pointing towards the attracting body. The differential forces $\frac{Mr}{\pi}\frac{f}{D^3}$ and $-\frac{Mr}{\pi}\frac{f}{D^3}$ are called Tidal Forces. As a planet can be considered to be made up of rings which are concentric on a line $\perp$ to the plane of our ring, it is evident that these tidal forces are continually deforming it into an oval which always points towards the attracting body. The moon and the sun both deform the earth into ovals pointing towards them. The longest axis of the oval is towards the attracting body, while the shortest axis is $\perp$ to the orbit.

The effect upon the rotation is similar to a friction band tending to slow the rotation down to coincidence with the revolutional period, and eventually this must occur. The rotation cannot fall below the revolution for in that case the tidal band would accelerate it into coincidence again. The effect in the case of the earth is very slight but the height of this solid tide may possibly approximate a foot. We do not know exactly what it is. It is constantly at work and it has very important and far-reaching effects which we shall discuss later. The earth is always rotating about two axes — one being the diurnal or polar axis, the other being the precessional axis which is $\perp$ to the former. The effect of the tidal brake-band is to oppose these two rotations, with the result in one case of lengthening the day and in the other of changing the inclination of the earth's axis. The oceans constitute a thin skin of water covering about ¾ of the earth's surface and they are of course subject to the tidal action we have just discussed. Theoretically, the effect of these ocean tides is to decrease the earth's motion, but compared with the solid tides of the earth's mass the effect is insignificant. Thus while the solid tides will undoubtedly bring the

rotation and revolution into coincidence in some finite time, the ocean tides could only effect this in aninfinite time. As our earth has not, and will not exist for an infinite time, such ocean effects are wholly negligible.

When two bodies are near each other, it will be seen that the terms containing $\frac{f}{D^3}$ and $\psi^2$ become very large, and the rending force may reach a limit beyond which the two bodies would be torn apart. Since these forces are also proportional to $r$, this limit would be reached sooner in larger bodies.

The particles of an attracted body, instead of executing circles about the rotation axis, are forced to move in ovals and as their distances from each other are now greater and now less, they are continually kneaded, stretched and compressed, and heat is produced, just as a hammered bar becomes hot. All this heat energy which is dissipated is produced at the expense of the rotation. When the rotational and revolutional periods coincide this action ceases. Such a coincidence of periods probably exists in all the satellites of our system, and in Venus and Mercury. Our moon is a near example.

## 15. Attractional and Molecular Rigidity

In a very large mass, such as the earth, there is no inherent elasticity of figure such as exists in small masses and which is due to the interaction of adjacent molecules. A small mass can have any shape while a large mass, of itself, has only one shape, viz., a sphere. It would be impossible for a body as large as the earth or the moon to have the shape of a bar. We must thus distinguish between two kinds of elasticity, or force tending to restore a body to its original shape after deformation.

Our conceptions of matter are formed from the conditions under which it exists at the earth's surface, but it is unsafe to project these ideas into places where the

conditions differ widely. A solid spring regains its shape because of the cohesion between its adjacent molecules and it possesses only a proximate molecular rigidity. As the size increases this kind of rigidity becomes of less importance until it is entirely overcome by self-gravitation and the mass flows into a sphere. A fluid is etymologically that which flows, so that in saying that the mass flows into a sphere we imply that it is no longer a solid body but a fluid body, although at the surface small portions may still be regarded as solid. When we melt a solid it loses largely its molecular rigidity and assumes a form of equilibrium depending chiefly upon gravitational forces. The matter in the interior of a large body, such as the earth, is thus certainly fluid or plastic because the shape it assumes is due solely to the enormous gravitational pressures and not at all to any molecular rigidity. The question as to whether the interior of the earth is solid or liquid is founded upon a confusion between general attractional rigidity and molecular rigidity. It is certainly plastic and very hot — probably well above 4000°C from the centre to a few miles from the surface. For all practical purposes, it is therefore in a molten condition.

A very large body must necessarily have a very high rigidity, exceeding that of a very small body, but the resistance to deformation is of a very different nature in the two cases and it is difficult to compare them. The earth has a high rigidity but it is manifestly improper to say that it has the rigidity of a globe of steel or of a globe of glass of equal size.

We may determine the rigidity of a glass marble, which is the rigidity of its proximate molecules, but if we increased it to the size of the earth, this particular kind of rigidity would become insignificant in comparison with its general attractional rigidity. A small piece of glass might have any shape, but a large piece could have only one shape. A further difficulty occurs in that a globe the size of the earth could not exist as glass or steel throughout.

The enormous condensation towards the centre would necessarily result in changing the kind of matter. We have previously referred to a suggestion that the ether may be the ground stuff out of which all the "elements" have been formed under varying conditions of pressure and temperature. This is merely a surmise, though perhaps a natural one when we see denser elements spontaneously splitting into lighter ones. We know that the average density of the earth is 5½ times that of water, while the surface density is only about 2¾. There must be much matter in the interior which is much denser than the surface matter and it seems probable that such matter may be like some of the denser elements with which we are familiar. A considerable part of the earth's interior is probably iron. Magnetic phenomena point to much paramagnetic matter — iron, nickel, cobalt. Siderites seem to be fragments of some former body, and their usual iron-nickel composition suggests that many bodies in the universe are automatically largely composed of these substances.

## 16. Density of the Earth

The matter of the earth was probably once homogeneous, but by self-gravitation has condensed to its present state. The density of this matter must, on the whole, increase regularly and progressively from the surface to the centre. What the exact law is, we do not know. Laplace assumed, as a hypothesis, that the increase of the square of the density was proportional to the increase of the pressure. That is to say the compression gradually decreases as the density increases. While perhaps not the actual law, it would seem that Laplace's formula must represent the actual conditions to a close approximation, even bearing in mind the possibility that the condensation may at certain stages have been per saltum, as where an element may have changed suddenly into a denser one.

Let $r$ be the distance of any level from the centre, $\rho$ and $f$ the density and acceleration at that level, and $\rho_a$ the

average density of all the matter *within* that level. Taking the direction outward as positive,

$$f = -\frac{1}{r^2}\int_o^r 4\pi\rho r^2 dr = -\frac{4\pi}{3} r\rho_a.$$

$$D_r f = -\frac{4\pi}{3}\left(\rho_a + rD_r\rho_a\right) = -\frac{2f}{r} - 4\pi\rho = 4\pi\left(\frac{2}{3}\rho_a - \rho\right) (1).$$

Hence $$rD_r\rho_a = 3(\rho - \rho_a) \qquad (2).$$

Laplace's formula is $\rho d\rho = kdp$, where $p$ is the pressure and $k$ is some constant. $dp = \rho f.dr$. Whence $D_r\rho = kf$ (3). $\rho$ is a function of $r$, or for any value of $r$ there is a single definite value for $\rho$.

$\rho = \varphi(r)$. By Maclaurin's theorem

$$\rho = \varphi(o) + \varphi^1(o)\, r + \varphi^{11}(o)\frac{r^2}{2!} + \text{etc.}$$

$$D_r\rho = \varphi^1(r) = kf. \quad \varphi^{11}(r) = kD_r f = 4\pi k\left(\frac{2}{3}\rho_a - \rho\right)$$

$$\varphi^{111}(r) = 4\pi k\left[\frac{2}{3}D_r\rho_a - kf\right] = 4\pi k\left[\frac{2(\rho - \rho_a)}{r} - kf\right]$$

$$\varphi^{iv}(r) = -\left[\frac{8\pi k\varphi^1(r)}{r} + \varphi^{11}(r) + \frac{4\varphi^{111}(r)}{r}\right].$$

$\varphi(o) = \rho_c$, where $\rho_c$ is the density at the centre.

$\varphi^1(o) = 0$, since the force at the centre is zero.

$\varphi^{11}(o) = -\frac{4\pi k}{3}\rho_c$. $\varphi^{111}(r)$ becomes indeterminate when $r$ is zero, but by differentiating numerator and denominator of the indeterminate term, we obtain zero as the limiting value.

$$\text{Limit}_{(r=o)}\ \varphi^{iv}(r) = -8\pi k\,\varphi^{11}_{(r=o)}(r) - 4\varphi^{iv}_{(r=o)}(r),$$

and $\underset{(r=o)}{\text{limit}}\ \varphi^{iv}(r) = \frac{35\,\pi^2k^2}{15}\rho_c$. Similarly we find that $\varphi^{v}(o) = 0$. All the odd orders of $\varphi(o)$ are zero and all the even orders are multiples of $\rho_c$ with increasing powers of $k$ in the coefficient. Thus $\varphi^{vi}(o)$ is a multiple of $-k^3\rho_c$, et c. $k$ is a very small quantity so that we can neglect higher

powers than the first. Hence we have

$$\rho = \rho_c\left[1 - \frac{4\pi k}{3}\cdot\frac{r^2}{2}\right] \qquad (4)$$

Denoting the average density of the body by $\underline{\rho_a}$,

$$\underline{\rho_a} = \frac{3\rho_c}{R^3}\int_0^R\left(1 - \frac{4\pi k}{3}\cdot\frac{r^2}{2}\right)r^2dr = \rho_c\left(1 - \frac{2\pi k}{5}R^2\right),$$

and
$$k = \frac{5}{2\pi R^2}\left(1 - \frac{\rho_a}{\rho_c}\right) \qquad (5),$$

and
$$\rho = \rho_c\left[1 - \frac{5}{3R^2}\left(1 - \frac{\rho_a}{\rho_c}\right)r^2\right] \qquad (6).$$

Hence, $5\underline{\rho_a} = 2\rho_c + 3\rho_s$ (7), where $\rho_s$ is the surface density. We have then the general law that for any large body (sphere), five times the average density is equal to twice the central density plus three times the surface density. In the case of the earth, many surface rocks have a density around 2.75. Assuming this as the average surface density, which is about half the average density of the whole earth, we have $\rho_c = \frac{7}{2}\rho_s$, or the density at the centre of the earth is 3½ times the surface density, or it is 9.63 times the density of water.

From Equa. (1) it will be seen that if the surface density of any body is less than ⅔ of its whole average density, there will be a level between the surface and the centre where $\frac{2}{3}\rho_a = \rho$, and $D_r f = 0$. At this point $f$ is a maximum and the $\rho$ curve has a point of inflexion. At this particular level the body exerts a greater attraction than at any other distance from the centre, and we shall call it the critical level. In the earth diagram, Fig. 8, this level is shown at $A$.

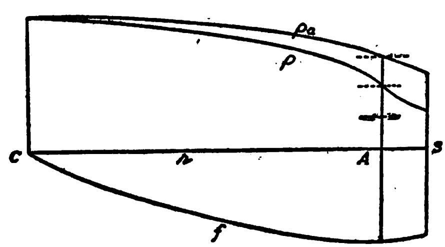

FIG. 8.

Let $x = AS = R - r$, where $r = CA$.

$$x = \frac{R}{2}\frac{(\rho_a - 2\rho_s)}{(\rho_a - \rho_s)}, \text{ and } \rho_a = \rho_c \left[\frac{\frac{5}{3}\frac{r^2}{R^2} - 1}{\frac{5}{3}\frac{r^2}{R^2} - \frac{2}{3}}\right].$$ Combining these equations we find that $r = .94R$. Or the critical level of the earth lies about 240 miles below the surface. The attraction therefore increases gradually up to this point after which it decreases regularly to zero at the centre. Only bodies of a certain size can have a critical level and the fact that this level is so near the earth's surface would seem to indicate that bodies of a little less size would have none. The moon probably has no critical level, or its attraction is a maximum at its surface, decreasing both towards and away from the centre.

Taking as our units, feet and seconds, the unit of mass will be that mass which attracts a like mass at unit distance with unit force, or a force giving it unit acceleration in unit time. The earth attracts unit mass at its surface with a force of 32 units. It would therefore attract unit mass at unit distance with $32R^2$ units of force, and unit mass is $\frac{M}{32R^2}$, where $M$ is the mass of the earth. The mass of the earth is equivalent $\frac{4\pi}{3}R^3 \times 5.5$ cu. ft. of water. Hence unit mass is equivalent to $.72R$ cu. ft. of water, where $R$ is expressed in feet. Or the matter of some 14 million cu. ft. of water concentrated into a point would attract a like mass at the distance of a foot with a force which would give it a velocity of a foot a second for every second that it acted. Unit density is this unit mass concentrated into a cubic foot. Hence, expressed in these units, the average density of the earth is $\frac{7.68}{R} = 3.68 \times 10^{-7}$, and the surface density is $\frac{3.84}{R}$. In the same way we find that the pressure at the centre of the earth is about 49 million pounds per square inch.

We shall refer here briefly to certain arguments for the solidity of the earth's interior. With the feeble pressures available in our laboratories it has been found that bodies which expand on melting have their melting points raised by pressure while bodies which contract on melting have their melting points lowered by pressure. It is argued that the enormous pressures in the interior will prevent the matter from liquefying. Further, certain surface rocks have been melted and it has been found experimentally that there is a slight expansion. On the other hand the molten lava lake of Kilauea is often skimmed over with a solid crust just as a lake of water is covered with a sheet of ice in winter, and in both cases the crusts are readily supported by the underlying liquid. This would indicate that the deeper matter, from which the lava comes, contracts on melting. Again iron contracts on melting and there is little doubt that iron forms a considerable part of the earth's interior. According to the argument, therefore, a very considerable part of the interior matter may have its melting point lowered by the enormous pressures and is therefore liquid. But such arguments are entirely inapplicable. Beyond a certain limit of pressure all matter becomes fluid, or flowing, and plastic. The interior of the earth is certainly plastic and very hot, and is therefore to all intents and purposes in a molten condition. The extrusion of molten rock from all parts of the earth's surface points strongly to such a general condition. Equally futile is the argument that the earth must be solid to have its high rigidity. The earth has a very high gravitational rigidity, but no molecular rigidity except at the surface. Surface conditions of matter cannot be projected into the interior.

## 17. Green's Theorem

Let us suppose that within a closed surface, $S$, $W$ is a function which has a single finite value for every point of our enclosed space, and varies continuously (without

abrupt change) in any direction. Choosing any axes, a line parallel to the $x$ axis must cut such a surface an even number of times.

$\int_1^2 \frac{\partial W}{\partial x} . \, dx = W_2 - W_1$, where $W_1$ is the value of $W$ at the point of entrance and $W_2$ its value at the point of exit. $\int\int\int \frac{\partial W}{\partial x} . \, dxdydz = \int\int Wdydz$, the volume integral on the left being taken throughout the whole closed space, and the surface integral over the whole surface. Representing the volume element, $dxdydz$, by $d\tau$ and the surface element $dydz$ by $dS \cos (nx)$, where $dS$ is the element of the closed surface cut out by an elementary parallelopiped and $(nx)$ is the angle between $x$ and the normal to the surface, drawn inward, $\int\int\int \frac{\partial W}{\partial x} d\tau = \int\int W \cos (nx) \, dS$. $W$ is any continuous finite point function, and if $U \frac{\partial V}{\partial x}$ be such a function, we can substitute it for $W$, or $\int\int\int \frac{\partial}{\partial x}\left(U \frac{\partial V}{\partial x}\right) d\tau$ $= \int\int U \frac{\partial V}{\partial x} \cos (nx) \, dS$ (1), with similar expressions for $y$ and $z$. Adding,

$$\int\int\int \left(\frac{\partial U}{\partial x} \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial V}{\partial y} + \frac{\partial U}{\partial z} \frac{\partial V}{\partial z}\right) d\tau =$$

$$- \int\int U \left(\frac{\partial V}{\partial x} \cos (nx) + \frac{\partial V}{\partial y} \cos (ny) + \frac{\partial V}{\partial z} \cos (nz)\right) dS$$

$$- \int\int\int U \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) d\tau. \quad (2)$$

This result is known as Green's Theorem. The derivatives here are partial as indicated by the notation and obviously $\frac{\partial V}{\partial x} \cos (nx) + \frac{\partial V}{\partial y} \cos (ny) + \frac{\partial V}{\partial z} \cos (nz) = \frac{\partial V}{\partial n}$, where $\frac{\partial V}{\partial n}$ signifies differentiation in the direction of the normal.

Hence we can write (2)

$$\iiint \left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z}\right) d\tau =$$

$$-\iint U \frac{\partial V}{\partial n}\, dS - \iiint U \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) d\tau.]$$

Since, by symmetry, $U$ and $V$ are interchangeable, we have also $\iint \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n}\right) dS =$

$$\iiint \left[V\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) - U\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) d\tau\right]$$

(3). Equa. (3) is known as Green's Theorem in its second form.

## 18. Gauss' Theorem

Since every atom is always in motion it radiates energy incessantly by communicating its motion to the ether. It is thus the centre of a field of force and the lines of force are straight lines radiating from this centre, while equipotential surfaces are spherical surfaces. The force at any point is $-\frac{m}{r^2}$, where $m$ is the mass of the radiating point. The integral of this with respect to $r$, in the direction of the force, is the work done, and the integral in the opposite direction is the work undone which is called the potential, or stored, energy, represented by $V$. Therefore $V = -W$. If we assume that attraction is effected through the agency of longitudinal waves in the ether, it can be shown (Mechanics of Electricity) that the force at any point is equal to the quantity of wave energy traversing (orthogonally) unit surface in unit time. In other words, the force is measured by the density of the flux of the energy at the point. The total energy traversing any surface surrounding the point is necessarily the same as that crossing any other such surface, or is equal to the flux of energy across unit sphere about the point. The density of the energy at any point is necessarily inversely as the surface, or as the square of the distance, so that our assumption that attraction is

effected by longitudinal waves contains implicitly the result, $f = -\frac{m}{r^2}$. The total flux of energy across any spherical surface is $\Sigma f = 4\pi m$, and this is the energy flux across any surface surrounding the point. It follows that for any number of points, or for any distribution of matter, the total flux across any surrounding surface is $4\pi M$. If the surface does not surround the radiating matter, since the energy cannot accumulate within this surface, but the amount of energy contained by the surface must at all times be constant, it follows that as much energy is always passing out of the surface as is entering it, so that the total flux of energy across its surface is zero. These results constitute Gauss' Theorem. Mathematically expressed,

$$\Sigma \frac{dV}{dn} dS = -\Sigma \frac{m}{r^2} \cos (nr)\, dS = 0,$$

when the matter is without the surface, and

$$\Sigma \frac{dV}{dn} dS = -\Sigma \frac{m}{r^2} \cos (nr)\, dS = 4\pi M,$$

when the matter is within the surface. The normal, $n$, is always supposed drawn inward from the surface. The potential due to a particle, $m$, at any point is the scalar (undirected) quantity, $V = \frac{m}{r}$, and the potential at that point due to any number of particles is $\Sigma \frac{m}{r}$.

## 19. Poisson's Equation

Selecting a certain point, let us surround it with a surface. This surface may cut matter, so that there is matter both within and without the surface. The total flux of energy through the surface from the outside matter is zero, while that from the inside matter is

$$\int\int \frac{dV}{dn} dS = -\int\int \frac{dm}{r^2} \cos (nr)\, dS = 4\pi M = 4\pi \int\int\int \rho d\tau$$

(1), $\rho$ being the density of the inside matter at any point.

From Green's theorem, putting $U = 1$,

$$\int\int \frac{dV}{dn}\, dS = -\int\int\int \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) d\tau \quad (2).$$

Designating the operation, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ by $\Delta$, and combining (2) with (1), we have

$\int\int\int \left(\Delta V + 4\pi\rho\right) d\tau = 0$ (3). If we allow our surface to close down upon our point, Equa. (3) will still be true. Therefore, $\Delta V + 4\pi\rho = 0$, which is Poisson's equation. It expresses the fact that if the potential at any point due to any distribution of matter, be operated upon by the operator $\Delta$, the result will be $-4\pi\rho$, where $\rho$ is the density of the matter at that point. If there is no matter at the point, $\Delta V = 0$, which is Laplace's equation.

## 20. Attraction of a Circular Disc

The attraction of a homogeneous circular disc upon unit mass situated in its axis, is $\int_0^R \frac{2\pi\sigma a r}{(r^2 + a^2)^{\frac{3}{2}}} dr$, where $R$ is the radius of the disc, $a$ the distance from the disc, and $\sigma$ the density of the matter upon it. Integrating,

$F = 2\pi\sigma\left(1 - \frac{a}{\sqrt{R^2 - a^2}}\right) = 2\pi\sigma\,(1 - \cos\alpha)$, where $\alpha$ is the angle subtended by $R$ at the attracted point. If the point is in the surface, $F = 2\pi\sigma$. In crossing the surface, since the attraction is $2\pi\sigma$ on one side and $-2\pi\sigma$ on the other side, there is a sudden change, or discontinuity, in the value of $F$ of $4\pi\sigma$. And in traversing any shell, no matter how matter is distributed over it, the change of the attraction must always be $4\pi\sigma$. For the attraction of an infinitely small disc at the point will be $2\pi\sigma$, and that of the rest of the shell could not have changed during the infinitely small motion of traversing the shell. We have already seen that the attraction of a spherical shell (homogeneous) changes from $4\pi\sigma$ just outside the shell to zero within the shell.

The attraction of an infinitely small disc of the shell at the point was $2\pi\sigma$. Hence the attraction of the rest of the shell was also $2\pi\sigma$. In traversing the shell the latter did not change, but the former changed to $-2\pi\sigma$. Hence the attraction within the shell became zero.

## 21. Maclaurin's Theorem

We have an ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, uniformly filled with matter of a density, $\frac{1}{4\pi}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$, and we wish to find if it is possible to distribute negative matter, or matter which repels instead of attracting, over its surface, in such a way that it will annul the external attractive field of the ellipsoid. For the purposes of analysis it would suffice to make use of an imaginary substance which we supposed to repel instead of attracting. Such a result would be obtained by simply making the density of such a substance negative instead of positive, as by making such changes in our formulas we reverse attractional actions. It is certain that these actions are effected through and by the ether, and probably by the agency of longitudinal waves. Assuming this agency, and calling matter denser than the ether, positive matter, and matter less dense than the ether, negative matter, it can be shown (Mechanics of Electricity) that positive matter attracts positive matter, while it repels negative matter. The repulsions of nature, of which there are many instances (comets' tails), as well as attractions, are probably to be explained in this way, and in all these cases the actual motions are accurately expressed analytically by simply making the density negative. The negative matter which we postulate in the present problem is therefore not merely an imaginary or mathematical conception, but probably represents an actual condition in nature.

But to return to our problem. If such a distribution of negative matter on the surface of the ellipsoid is possible,

then there will be no flux of energy through this surface, and the potential at the surface and at all external points must be zero. Gauss' theorem requires that the positive and negative matter within the surface shall be equal, or that the algebraical sum of this matter shall be zero. The potential at any point, $x$, $y$, $z$, within the surface must be, $V = \frac{1}{2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)$, for this makes the potential zero at the surface, and at any interior point $\rho = -\frac{1}{4\pi}\Delta V$, which satisfies Poisson's equation.

The distribution of matter over a surface, without any thickness, like the concentration of matter in a point, is a purely imaginary mathematical conception, but if we have a shell bounded by two surfaces which though infinitely near give the shell varying thicknesses at different points, and we fill this shell with homogeneous matter of any density we please, then we have a real distribution.

Since the surface of the ellipsoid is an equipotential surface, the force inward from this surface must be everywhere normal, and this force is

$$\frac{\partial V}{\partial n} = \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right]^{\frac{1}{2}} = \left[\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right]^{\frac{1}{2}}.$$

Since in crossing the surface the force must change abruptly by an amount $4\pi\sigma$, where $\sigma$ is the density of the surface matter at the point of crossing, and since the length, $p$, of the perpendicular from the centre to the tangent plane at any point is

$$\left[\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right]^{-\frac{1}{2}}, \text{ we have } \frac{\partial V}{\partial n} = -4\pi\sigma = \frac{1}{p}.$$

Hence if we distribute negative matter over the surface in such a way that at every point its density is $-\frac{1}{4\pi}\cdot\frac{1}{p}$, the problem is solved.

The equation of an ellipsoid in terms of the $\perp$, $p$, from the centre to a tangent plane at any point, and

$\alpha$, $\beta$, $\gamma$, the direction angles of this $\perp$ referred to the principal axes, $a$, $b$, $c$, is $p^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma$. For projecting the co-ordinates of the point on the $\perp$, we have $p = x \cos \alpha + y \cos \beta + z \cos \gamma$, and the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Varying only $x$ and $y$ in these equations we have $dx \cos \alpha + dy \cos \beta = 0$, and $\frac{xdx}{a^2} + \frac{ydy}{b^2} = 0$. Hence $\frac{b}{a}x \cos \beta = \frac{a}{b}y \cos \alpha$ with similar expressions for $x$ and $z$, and $y$ and $z$. $p^2 = x^2 \cos^2 \alpha + y^2 \cos^2 \beta + z^2 \cos^2 \gamma + 2xy \cos \alpha \cos \beta + 2xz \cos \alpha \cos \gamma + 2yz \cos \beta \cos \gamma = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma -$

$$\left(\frac{b}{a}x \cos \beta - \frac{a}{b}y \cos \alpha\right)^2 - \left(\frac{c}{a}x \cos \gamma - \frac{a}{c}z \cos \alpha\right)^2$$
$$- \left(\frac{c}{b}y \cos \gamma - \frac{b}{c}z \cos \beta\right)^2.$$

The last three terms are zero.

If we vary the lengths of the axes of an ellipsoid but keep the direction of the $\perp$ to a tangent plane unaltered, the variation of the length of $p$ will measure the thickness at the point of tangency of the shell formed by the original surface and the varied surface.

$$\delta p = \frac{\cos^2 \alpha \cdot \delta a^2 + \cos^2 \beta\, \delta b^2 + \cos^2 \gamma\, \delta c^2}{2p}.$$

In order that the thickness of the shell, or amount of matter in it at every point, shall be proportional to $\frac{1}{p}$, we must vary the surface so that $\delta a^2 = \delta b^2 = \delta c^2$.

But this is a property of confocal ellipsoids, for the condition of confocality is $a^2 - b^2 =$ Const. $b^2 - c^2 =$ Const., or $\delta a^2 = \delta b^2 = \delta c^2$.

Calling the thickness of the shell at any point, $t$, and the common variation of the squares of the semi-axes, $q$, we have $- \sigma = \frac{1}{4\pi} \cdot \frac{2t}{q}$.

If then we have an infinitely thin shell, bounded by two confocal ellipsoids, filled with negative matter, and the

inner ellipsoid filled with an equal amount of positive matter, both matters being homogeneous, there will be no external field. The field of the confocal shell is therefore the same, but reversed, as that of a homogeneous confocal ellipsoid of the same mass, and if the masses are not the same the contours of the two fields, or lines of force, are the same, but the intensities of the forces at different points are proportional to the masses.

By increasing or decreasing our original ellipsoid through the addition or subtraction of infinitesimal confocal shells, but keeping always the same amount of homogeneous matter filling the ellipsoid, we shall not alter the external field. Hence any confocal shell has the same external field as any other confocal shell with the same foci and mass, or as any confocal homogeneous ellipsoid of the same mass. If the masses are not equal, the direction of the force at all external points is the same, but the intensity is proportional to the mass. This is Maclaurin's Theorem. Generally, all thick or thin homogeneous shells bounded by confocal ellipsoids have the same external fields if their masses are equal: if not, the forces at any point are in the same direction and proportional to the masses.

Since in the limiting case of confocal ellipsoids, the ellipsoid becomes an elliptic disc with semi-axes $\sqrt{a^2 - c^2}$ and $\sqrt{b^2 - c^2}$, an elliptic disc will attract the same as any confocal ellipsoid, if we distribute uniformly upon the disc an equal amount of matter. And a circular disc will exert the same attraction on all external points as an ellipsoid of revolution of equal mass, provided the radius of the disc is $\sqrt{a^2 - b^2}$ and it is similarly placed.

## 22. Theorem on Attraction

The resultant attraction of a circular disc on a point in its axis we have seen to be $2\pi\sigma\ (1 - \cos\alpha)$, (Art. 20). $2\pi(1 - \cos\alpha)$ is the solid angle subtended by the disc at the point. For if $r$ is the radius of the sphere which con-

tains the circumference of the disc and has the point for its centre, the surface of the spherical cap cut out by the disc is $2\pi r.r(1 - \cos\alpha)$. The attraction of the disc is therefore measured by the solid angle, or surface of unit sphere cut out by the cone formed by the disc. Generally the resultant attraction of a homogeneous disc of any shape in the direction $OP$, the $\perp$ from the point to the disc, is measured by the surface cut out of unit sphere, drawn about the point, by the cone having the disc as a base and the point as an apex. An elementary cone $d\omega$ cuts out of a spherical surface through $A$, an element $dS$, and an element $\dfrac{dS}{\cos\alpha}$ out of the disc. The resolved part of the attraction of the element in the direction $OP$ is $\dfrac{dS}{\rho^2}$, $\rho$ being the distance of the element.

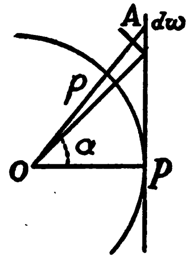


FIG. 9.

$\Sigma \dfrac{dS}{\rho^2}$ is the solid angle subtended by the disc at $O$, and the theorem is proved. When the disc is symmetrical about the $\perp$, this measures the total resultant attraction. The theorem gives a means of finding the attraction of an ellipsoid on an external point, in the direction $\perp$ to its focal, or $a$, $b$, plane. For we can reduce the ellipsoid to an elliptic disc confocal with the ellipsoid and of equal mass, and the attraction of this disc, $\perp$ to its plane, is the surface cut out of unit sphere about the point by the cone of the disc, into its density.

## 23. Homoeoids

For simplicity, we shall define a homoeoid as an infinitesimal shell bounded by two similar ellipsoids. The condition of similarity between two ellipsoids is

$$\frac{x^2}{a^2(1+q)} + \frac{y^2}{b^2(1+q)} + \frac{z^2}{c^2(1+q)} = 1,$$

and we have seen that the condition of confocality is $\frac{x^2}{a^2+q} + \frac{y^2}{b^2+q} + \frac{z^2}{c^2+q} = 1$. That is, the variation of $a^2$, $b^2$, $c^2$, is $qa^2$ etc., in the first case and $q$ in the second. $p = \sqrt{a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma}$, and the thickness, $t$, of a shell at any point is the variation of the constantly directed $\perp$, $p$, to a tangent plane at that point, as we vary the axes infinitesimally.

$\delta p = t = \frac{\cos^2\alpha\delta a^2 + \cos^2\beta\delta b^2 + \cos^2\gamma\delta c^2}{2p}$. If the ellipsoids are similar, $\delta p = t = \frac{qp}{2}$; if they are confocal, $\delta p = t = \frac{q}{2p}$. Hence the density of matter at any point of a homoeoid varies directly as $p$, while in a confocal shell it varies inversely as $p$.

Let us describe an infinitesimal double cone, $d\omega$, having some point, $P$, in the interior of a homoeoid as its common apex. These cones will cut out elements, $dS.dp$, attracting the point with a force $\frac{dS.\,dp}{r^2}$, where $r$ is the distance of $P$ from the element. $p = r\sin\tau$, where $\tau$ is the angle between $r$ and the tangent plane at $dS$. $dp = \sin\tau dr$, and the attraction of each element is $\frac{\sin\tau\, dr\, dS}{r^2} = drd\omega$, since $\frac{\sin\tau\, dS}{r^2} = d\omega$. Now any plane section of a homoeoid must be two similar ellipses, and passing a plane through the axis of our cones, by varying its direction, it will be possible to make this axis a diameter of the two similar ellipses. Hence generally whenever a straight line is passed through a space bounded by two similar and similarly placed ellipsoids, the two intercepts of the line between the two surfaces will be equal. It follows that there is no force in the interior of a homoeoid, whether the shell is thick or thin. It may be observed that a charge of electricity distributes itself over the surface of an

ellipsoidal conductor homoeoidally since there is no force in the interior, or the density of the electricity at any point is directly proportional to the $\perp$ on the tangent plane.

A homoeoid is obviously an equipotential surface for its own field, for the potential at all interior points and at its surface is constant. If we draw an equipotential surface just outside and infinitely near, the distance between the two surfaces will be inversely as the density $\sigma$ at any point, for $F = \frac{dV}{dn}\cdot$ The density is proportional to $p$ and the thickness of the shell formed by the two surfaces is as $\frac{1}{p}\cdot$ Hence the infinitely near equipotential surface is an ellipsoid confocal with the homoeoid. If we distribute homoeoidally on the confocal ellipsoid a mass equal to the original mass, it will be an equipotential surface for its own field and the flux leaving this surface will be the same as the flux from the original mass. Hence if we distribute homoeoidally equal masses on two infinitely near confocal ellipsoids their external fields will be the same. By a simple extension of the reasoning it is evident that any two confocal ellipsoids on which equal masses are distributed homoeoidally have identical external fields, and if the masses are not equal, the force at any external point is in the same direction and proportional to the mass. Further, all external equipotential surfaces of a homoeoidal distribution are confocal ellipsoids. This is Charles' Theorem.

### 24. The Potential Function

Fig. 10 represents any two masses with centres of inertia at $G$ and $G'$. $P$ and $P'$ are any two points in the masses and $GG' = R$, $GP = r$, $G'P = \rho$, $GP' = \rho'$, and $PP' = \rho''$. The two bodies have a mutual field of energy represented by their mutual potential which for any two elements is $\frac{dm\, dm'}{\rho''}$. We wish to find the sum of the po-

tentials for all the elements, thus deriving the total energy of the system. The potential between the first body and an element $dm'$ of the second body at $P'$ is $dm'\,\Sigma\,\dfrac{dm}{\rho''}$, and the total energy is $\Sigma\left(dm'\,\Sigma\,\dfrac{dm}{\rho''}\right)$. Taking rectangular co-ordinates with $G$ as origin and $\rho'$ as axis of $x$, we have, since $\rho''^2 = \rho'^2 - (2\rho' x - r^2)$,

$$\Sigma\,\frac{dm}{\rho''} = \Sigma\,\frac{dm}{\rho'}\left(\frac{2\,\rho' x - r^2}{\rho'^2}\right)^{-\frac{1}{2}} = \Sigma\,\frac{dm}{\rho'}\left(1 + \frac{x}{\rho'} + \frac{3x^2 - r^2}{2\,\rho'^2} + \frac{5x^3 - 3xr^2}{2\rho'^3} + \frac{35x^4 - 30x^2r^2 + 3r^4}{8\rho'^4} + \right)$$

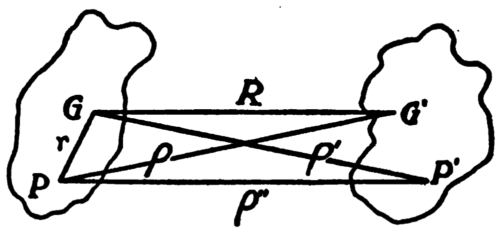


FIG. 10.

Since $G$ is the centre of inertia all the odd powers vanish in the integration, and we have

$$\Sigma\,\frac{dm}{\rho''} = \Sigma\,\frac{dm}{\rho'}\left(1 + \frac{3x^2 - r^2}{2\rho'^2} + \frac{35x^4 - 30x^2r^2 + 3r^4}{8\rho'^4} + \right).$$

$\Sigma\, dmr^n$ is evidently a constant. $\Sigma\, dmr^2 = \dfrac{A + B + C}{2}$.

$\Sigma\, dmx^2 = \dfrac{A + B + C}{2} - I\rho'$, where $I\rho'$ is the moment of inertia about $\rho'$ as an axis.

The second term is therefore $\dfrac{A + B + C - 3I\rho'}{2\rho'^2}$. Let us suppose that the bodies are spheroids, either homogeneous or made up of homogeneous confocal shells. Integrating we have

$$\Sigma\, dmx^4 = \frac{15}{7M}\left[\frac{A + B + C}{2} - I\rho'\right]^2.$$

In like manner

$$\Sigma\, dmx^2r^2 = \Sigma\, dmx^4 + \frac{5}{7M}\left[\frac{A+B+C}{2} - I\rho'\right] I\rho' +$$
$$\frac{2M}{35}(a^2 - b^2)^2 \left[\frac{I\rho' - A}{C - A} - \left(\frac{I\rho' - A}{C - A}\right)^2\right],$$

where $a$ and $b$ are the semi-axes of the shells. The third term therefore contains $I\rho_1{}^2$, $I\rho_1$, $I\rho'^0$, with determinable coefficients in the numerator and $8\rho'^4$ in the denominator. Every term of the order $2n$ will have $I\rho_1{}^n$, $I\rho_1{}^{n-1}$, . . . $I\rho'^0$, in the numerator and $\rho'^{(2n+1)}$ in the denominator. Hence we can write

$$\Sigma \frac{dm}{\rho''} = \frac{M}{\rho'} + F\left(\frac{I\rho'}{\rho'}\right),$$

where $F(\;)$ is an algebraical series having the moment of inertia, $I\rho$, about an axis $\rho$ in ascending powers in the numerators, and $\rho$ in ascending powers in the denominators. The total potential is

$$V = \Sigma\left(dm'\,\Sigma\frac{dm}{\rho''}\right) = \Sigma\, dm'\left[\frac{M}{\rho'} + F\left(\frac{I\rho'}{\rho'}\right)\right].$$

Since $$\Sigma\frac{dm'}{\rho'} = \frac{M'}{R} + F\left(\frac{I'}{R}\right),$$

where $I'$ is the moment of inertia of the second body about $R$ as an axis,

$$V = \frac{MM'}{R} + M'F\left(\frac{I}{R}\right) + \Sigma\, dm\, F\left(\frac{I'\rho}{\rho}\right) = \frac{MM'}{R} +$$
$$MF\left(\frac{I'}{R}\right) + \Sigma\, dm'\, F\left(\frac{I\rho'}{\rho'}\right).$$

Here $I$ and $I\rho'$ are the moments of inertia of the first body about the axes $R$ and $\rho'$, and the primes are the moments of inertia of the second body.

$\Sigma\, dm'\, F\left(\frac{I\rho'}{\rho'}\right)$ is an integral equivalent to the total mass $M'$ concentrated at some point in the second body at a distance $\underline{\rho}'$ from $G$, into the average of the function $F\left(\frac{I\rho'}{\rho'}\right)$. It is obvious that $\underline{\rho}'$ must coincide with $R$, for

otherwise the forces giving the translational motions to the two centres of inertia would not be equal and opposite (in the same line), and by their mutuality this must be the case. Hence we can write

$$V = \frac{MM'}{R} + M'F\left(\frac{I}{R}\right) + MF\left(\frac{I'}{R}\right)$$

(1), where $F\left(\frac{I}{R}\right) = \frac{kI + k}{R^3} + \frac{kI^2 + kI + k}{R^5} + \text{etc.},$

$k$ representing certain determinable constants. As has been already shown, the forces acting upon the first body are equivalent to a single force acting upon the mass $M$ supposed concentrated into its centre of inertia, together with a couple acting about an axis through the centre of inertia. Likewise the forces acting upon the second body are equivalent to a single force acting upon the mass $M'$ concentrated into its centre of inertia, together with a couple about an axis through this centre. From their mutuality the two translational forces are equal and opposite and the two couples are equal and opposite, or their axes are in opposite directions, since a couple is represented by a vector. The force in any direction is the rate at which the potential energy is used up in that direction, or the derivative of the potential with respect to the direction. The force urging the centres of inertia together is $\frac{dV}{dR}\cdot$ If we simply turn the bodies about $R$, the potential is not altered.

Considering the second body simply as a material point of mass $M'$, Equa. (1) becomes

$$V = \frac{MM'}{R} + M'F\left(\frac{I}{R}\right). \tag{2}$$

It will be noted that the attraction in the plane of the equator of a spheroid is greater than if the mass were concentrated into its centre, while the attraction in its axis is less, or for equal distances the attraction is a maximum in the plane of the equator and a minimum in the axis. It will also be noted that the motion of the centre

of inertia is the same whether we apply the forces parallel to their original directions to this centre, or subject the mass concentrated into this centre to the field. In general this is not the case. [See Art. 8.]

If $\gamma$ is the angle which $R$ makes with the equator of the spheroid, then the couple tending to bring the equatorial plane into coincidence with $R$ is $\frac{dV}{d\gamma} = -\frac{3M'}{2R^3}\frac{dI}{d\gamma} = \frac{3M'}{R^3}(C - A)\sin\gamma\cos\gamma$, since $I = C\cos^2\gamma + A\sin^2\gamma$, to a close approximation.

### 25. Rotary Motion

A disc spins (Fig. 11) on its axis $PP'$, which is held in a ring which can turn about a horizontal axis $HH'$, and the whole turns about a vertical axis $VV'$.

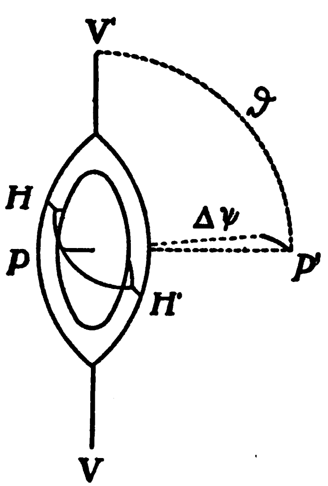


FIG. 11.

If the disc is not rotating and we turn it about the vertical axis, it will simply obey the turn and its axis $PP'$ will remain in the horizontal plane. If it is rotating and we turn the axis through a horizontal angle $d\psi$ in the time $dt$, the axis will not remain horizontal. Let $\omega$ be the angular velocity with which the disc spins. This velocity cannot be influenced by the turn since there is no couple about $PP'$, and we further suppose no friction.

Let $C$ be the moment of inertia about the axis of the disc, and $A$ that about an axis in its plane. We can resolve the original moment of momentum, $C\omega$, into the two components $C\omega\cos d\psi$ and $C\omega\sin d\psi$ in the horizontal

plane. Since no motion can cease instantly, when we turn the axis of the disc through the angle $d\psi$ in the infinitesimal time, $dt$, the above moments will persist, and the moment of momentum about the axis will be $C\omega \cos d\psi$ and there will also be a moment $C\omega \sin d\psi$ about a $\perp$ axis in the horizontal plane. The rate of change of a moment of momentum measures a couple about its axis. The rate of change of the moment about the axis of the disc is $\frac{C\omega \cos d\psi - C\omega}{dt}$, and the rate of change about the $\perp$ axis is $\frac{C\omega \sin d\psi}{dt}$. At the limit the first becomes zero, and the second is $C\omega\dot{\psi}$. Hence at the beginning of the turn a couple is set up tending to bring the axis of the disc into coincidence with the axis of the turn, and if the turn is continued these two axes will coincide both in direction and sense of motion about them. Generally, whenever a rotating mass is turned about an axis $\perp$ to its rotation axis, a couple is set up about an axis $\perp$ to the former two axes, and this is called the gyroscopic couple. Any rotating mass is a gyroscope. If $\dot{\vartheta}$ is the angular acceleration about the gyroscopic axis, $C\omega\dot{\psi} = -A\dot{\vartheta}$ (1). This is the fundamental gyroscopic law from which all the properties of rotary motion can be derived. The gyroscopic couple is an internal force due to the inertia of the particles of the disc and is nothing more than the moments of the centrifugal forces which arise during the turn.

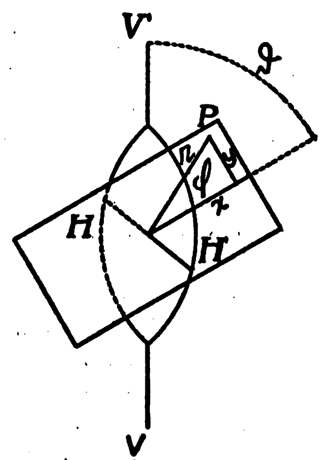

FIG. 12.

In Fig. 12 a lamina can turn about a horizontal axis $HH'$ and this axis is turned about a vertical axis, $VV'$, with angular velocity $\dot{\psi}$. A point $P$ has co-ordinates, $x$, $y$, referred to axes through the centre of the lamina, and the

radius $r$ makes an angle $\varphi$ with the axis of $x$. The centrifugal force of an element $dm$ at $P$, is $dmr \sin (\vartheta - \varphi)\dot{\psi}^2$. The moment of this force about the horizontal axis is

$$dmr \sin (\vartheta - \varphi) \cos (\vartheta - \varphi) \dot{\psi}^2.$$

Since $$\frac{x}{r} = \cos \varphi \text{ and } \frac{y}{r} = \sin \varphi,$$

the total moment is

$$\Sigma dm \sin \vartheta \cos \vartheta (x^2 - y^2)\dot{\psi}^2 - \Sigma dm (\cos^2\vartheta - \sin^2\vartheta)xy\dot{\psi}^2.$$

From symmetry, $\Sigma dmxy$ vanishes, and we have the well known result,

$$\Sigma dm \sin \vartheta \cos \vartheta (x^2 - y^2)\dot{\psi}^2 = (C - A) \sin \vartheta \cos \vartheta\dot{\psi}^2,$$

where $C$ is the moment about the $y$ axis and $A$ that about the $x$ axis.

But we can get this result at once by gyroscopic principles. Resolving $\dot{\psi}$ into an angular velocity $\dot{\psi} \sin \vartheta$ about the $y$ axis and $\dot{\psi} \cos \vartheta$ about the $x$ axis, we see that these are two gyroscopic couples about $HH'$, viz., $C \sin \vartheta\dot{\psi}. \cos \vartheta\dot{\psi}$ and $A\dot{\psi} \cos \vartheta. \dot{\psi} \sin \vartheta$, opposing each other. The net result is $(C - A) \sin \vartheta \cos \vartheta\dot{\psi}^2$, tending to place the $x$ axis horizontally. The gyroscopic couple is therefore merely the integral of all the centrifugal moments.

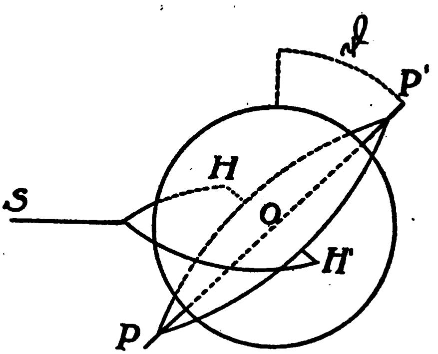

FIG. 13.

As another example let us suppose a sphere, Fig. 13, rotating about an axis $PP'$ with angular velocity $\omega$, while $PP'$ can turn about a horizontal axis $HH'$, which is held by a fork which turns about a fixed centre at $S$ in a horizontal plane, with angular velocity $\dot{\psi}$.

Resolve the rotation $\omega$ into $\omega \cos \vartheta$ about a vertical axis and $\omega \sin \vartheta$ about $SO$. Let $SO = D$. Considering a ring of matter about $SO$, the horizontal velocity of any particle

in this ring is $D\dot{\psi} - r\omega \sin \vartheta \cos \varphi$, where $\varphi$ is the angle any radius makes with the vertical. The centrifugal force is $\frac{dm}{D}\,[D\dot{\psi} - r\omega \sin \vartheta \cos \varphi]^2$ and the moment of this force about $HH'$ is $\frac{dm}{D}\left[D\dot{\psi} - r\omega \sin \vartheta \cos \varphi\right]^2 r \cos \varphi$. The sum of the moments is $\int_0^{2\pi} \frac{r}{D}\left[D\dot{\psi} - r\omega \sin \vartheta \cos \varphi\right]^2$ $r \cos \varphi d\varphi = 2\pi r . r^2\omega \sin \vartheta\dot{\psi} = mr^2\omega \sin \vartheta\dot{\psi}$. The moment of momentum of the ring about $SO$ is $mr^2\omega \sin \vartheta$ and $\dot{\psi}$ is the angular velocity with which the momental axis turns. Hence the integral of the centrifugal moments about $HH'$ is simply the gyroscopic couple.

For a disc the total centrifugal moments about $HH'$ are $\sum\limits_0^R 2\pi r^3\omega \sin \vartheta\dot{\psi} dr = \frac{\pi R^4}{2} \cdot \omega \sin \vartheta\dot{\psi}$. But $\pi R^2$ is the mass of the disc and $\frac{R^2}{2}$ is $k^2$, where $k$ is the radius of gyration, and the integral gives simply the gyroscopic couple.

For the whole sphere, using the relation $r^2 = R^2 - k^2$, where $k$ is the distance of any vertical disc from the centre, $\sum\limits_0^R \pi r^4 \omega \sin \vartheta\dot{\psi}\, dk = \frac{4\pi R^3}{3} \cdot \frac{2R^2}{5} \cdot \omega \sin \vartheta\dot{\psi}$. $\frac{4\pi R^3}{3}$ is the mass and $\frac{2R^2}{5} = k^2$, and we see again the identity of the gyroscopic couple with the sum of the centrifugal moments. The gyroscopic principle gives at once the total centrifugal moments acting about any axis. Without its use many rotational problems would be practically insoluble.

## 26. Euler's Dynamical Equations

Let us suppose that a triaxial body is given an impulsive velocity about some axis through its centre of inertia, which is fixed. The resulting motion will in general be unstable. For we can resolve the impulsive velocity into

$\omega_1$, $\omega_2$, $\omega_3$ about the three principal axes and it is evident that a pair of gyroscopic couples will result about each axis. With due regard to signs, we can write

$$(B - C)\,\omega_2\omega_3 = A\dot{\omega}_1$$
$$(C - A)\,\omega_1\omega_3 = B\dot{\omega}_2$$
$$(A - B)\,\omega_1\omega_2 = C\dot{\omega}_3$$

These are Euler's dynamical equations. The mutual interaction of these couples will cause the original instaneous axis to shift continually.

If, however, the original velocity were imparted about a principal axis, the motion would be stable, for there would be no couples.

Let us suppose a uniaxial body to be rotating about some axis which is struck a sharp blow in any direction. We always consider the centre of inertia as fixed. If the body were not rotating it would simply turn about an axis $\perp$ to the blow. But rotating, the instantaneous velocity imparted by the blow combines with the original rotation, and we have a new rotation about an axis which is in the plane of the other two. The rotation axis, instead of moving in the direction of the blow, in fact moves in a direction $\perp$ to it. If $\dot{\varphi}$ be the impulsive velocity imparted about an axis $\perp$ to the rotation axis and $i$ the angle the new resultant axis makes with the original axis, $\tan i = \frac{\dot{\varphi}}{\omega}$, and the resultant angular velocity is $\sqrt{\dot{\varphi}^2 + \omega^2}$. Since all axes are principal axes, the new rotation will be stable.

### 27. Biaxial Bodies Under No Forces

If we subject a biaxial body to an impulsive couple, measured by $G$, about an axis making an angle $\vartheta$ with the $C$ axis, or axis of unequal moment, this is equivalent to an impulsive moment about the $C$ axis, together with an impulsive component about a $\perp$ axis in the $GC$ plane. The instantaneous axis will lie in this plane. If $\omega_i$ is the instantaneous angular velocity and $i$ the angle which the instantaneous axis makes with $C$, while $\vartheta$ is the angle $G$

makes with $C$, $G \cos \vartheta = C\omega_i \cos i$, and $G \sin \vartheta = A\omega_i \sin i$. Hence $A \tan i = C \tan \vartheta$ (1), and $i$ and $\omega_i$ are determined. $G$ is represented by a vector, constant in amount and direction, and is called the Invariable Line. Since there can be no rotation or moment about an axis $\perp$ to $G$, the $C$ axis and the instantaneous axis, which always lie in the $GC$ plane, cannot change their inclinations to the invariable line. The motion therefore will consist of a rotation of the $GCi$ plane around the invariable line. Let $\dot\psi$ be the angular velocity of this plane. Then the $C$ axis turns about a $\perp$ axis in this plane with angular velocity $\dot\psi \sin \vartheta = \omega_i \sin i$, and the angular velocity about the $C$ axis is $\omega_i \cos i = \omega$. It is readily seen that the gyroscopic couples about an axis $\perp$ to the $GC$ plane exactly balance, so that there can be no motion about such an axis. For the gyroscopic couples about this axis are $C\omega\dot\psi \sin \vartheta$ and $A\dot\psi \sin \vartheta . \dot\psi \cos \vartheta$, and by (1) these are equal and opposite. The motion is thus completely determined. It consists of a steady rotation of the $CGi$ plane about the invariable line, which is the axis of the impulsive couple, and the unequal, or $C$, axis, and the instantaneous axis are fixed in this plane. $G$, $A$, $C$ and $\vartheta$ are given, and from these $\omega$, $\omega_i$, $i$ and $\dot\psi$ are readily found. The motion of the unequal axis about the invariable line is called the Precession, and $\dot\psi$ is the precessional velocity.

## 28. Triaxial Body Under No Forces

When a triaxial body is subjected to an impulsive couple about an axis through its centre of inertia, the case becomes more complicated. If $A$, $B$, $C$ are the principal moments of inertia of such a body, in ascending order, then $Ax^2 + By^2 + Cz^2 = 1$ is the momental ellipsoid of the body, and we have seen that it has the property that the square of the reciprocal of any of its radii is equal to the moment of inertia of the body about that radius.

Draw a radius, $r$, to any point on the momental ellipsoid and on the plane tangent at this point drop a $\perp$, $p$,

from the centre. The equation of the momental ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = Ax^2 + By^2 + Cz^2 = 1 \qquad (1)$$

Whence $\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^2} = A^2x^2 + B^2y^2 + C^2z^2 =$ $r^2 (A^2 \cos^2\alpha^1 + B^2 \cos^2\beta^1 + C^2 \cos^2\gamma^1)$ (2), where $\alpha^1$, $\beta^1$, $\gamma^1$ are the direction angles of a radius referred to the principal axes. If now we apply an impulsive couple, $G$, having $p$ as its axis, the moment of momentum of the body about this particular $p$ must remain constant throughout the motion and equal to $G$. Resolved into its components about the principal axes, $G \cos \alpha = A\omega_1$, $G \cos \beta = B\omega_2$, $G \cos \gamma = C\omega_3$, where $\alpha$, $\beta$, $\gamma$ are the direction angles of $p$ referred to the principal axes, and $\omega_1$, $\omega_2$, $\omega_3$ are the angular velocities about these axes at any instant. Hence, $G^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2$ (3). Also, since the kinetic energy, $T$, must remain constant, $A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2T = I\omega_i^2 = \frac{\omega_i^2}{r^2}$ (4), where $I$ and $\omega_i$ are the moment of inertia and the angular velocity about the instantaneous axis. Hence $\omega_i = \sqrt{2T}.r$, or the instantaneous velocity is proportional to the radius of the instantaneous axis.

$$\text{Since } \frac{\omega_1}{\omega_i} = \cos\alpha^1, \ \frac{\omega^2}{\omega_i} = \cos\beta^1, \ \frac{\omega^3}{\omega_i} = \cos\gamma^1 \qquad (5)$$

$$\text{and} \quad r^2 = \frac{\omega_i^2}{2T}, \frac{1}{p^2} = \frac{A^2\omega_1^2 + B^2\omega^2{}_2 + C^2\omega_3^2}{2T} = \frac{G^2}{2T} \quad (6)$$

Hence the end of the instantaneous axis will always be in the original tangent plane, which is therefore fixed and called the Invariable Plane. The perpendicular, $p$, is constant in length and direction and is called the Invariable Line. The angular velocity of the body about $p$ is $\omega_i \frac{p}{r} = \frac{2T}{G}$, and it is therefore constant. If, then, we suppose the momental ellipsoid to roll on the invariable plane preserving a constant angular velocity about $p$, the motion of the body will be exactly represented.

The invariable line and the instantaneous axis cut out

cones in the body during the motion, and these are called the invariable and instantaneous cones. Their equations are found from (3) and (4), which combined are

$$(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)\ G^2 = (A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2)\ 2T.$$

Taking co-ordinates, $x$, $y$, $z$, proportional to the direction cosines of either the invariable line or the instantaneous axis, and eliminating $\omega_1$, $\omega_2$, $\omega_3$ by the relations $G \cos \alpha = A\omega_1 = kGx$, etc., for the invariable line, and by the relations $x = k\alpha^1 = k\dfrac{\omega_1}{\omega_i}$, etc., for the instantaneous axis, we have as the equations of the invariable and instantaneous cones respectively,

$$\frac{2AT - G^2}{A} x^2 + \frac{2BT - G^2}{B} y^2 + \frac{2CT - G^2}{C} z^2 = 0 \quad (7)$$

$$A\,(2AT - G^2)\,x^2 + B\,(2BT - G^2)\,y^2 + C\,(2CT - G^2)\,z^2 = 0 \quad (8)$$

When $2AT$, or $2CT$, equals $G^2$, these cones, which are of the second degree, become two imaginary planes, which however intersect in a real line. From (1) and (5) this condition is that $p = a$, or $p = c$, or the instantaneous axis coincides with the major or the minor axis of the momental ellipsoid at the beginning of the motion. If $2BT = G^2$, the cones reduce to two real planes. Here $p = b$, or the $\perp$ is equal to the middle axis of the ellipsoid.

The instantaneous cone intersects the surface of the ellipsoid in a curve which is called the Polhode. This curve is evidently the locus of all those points on the ellipsoid for which $p$ has a constant value. Its equation is found by combining the equation of the ellipsoid with another expressing the fact that $p$ remains constant. Hence it is

$$A\left(\frac{1}{p^2} - A\right) x^2 + B\left(\frac{1}{p^2} - B\right) y^2 + C\left(\frac{1}{p^2} - C\right) z^2 = 0,$$

since $$\frac{1}{p^2} = A^2x^2 + B^2y^2 + C^2z^2 = \frac{G^2}{2T}.$$

This is the equation of a cone of the 2nd degree with its apex at the centre. If $p$ is equal to the middle axis, or $\dfrac{1}{p^2} = B$, we have $A\,(B - A)\,x^2 = C\,(C - B)\,z^2$, which represents

two planes intersecting in the middle axis and making angles with the $A$, $B$, plane whose tangents are

$$\pm\sqrt{\frac{A\ (B - A)}{C\ (C - B)}}.$$

Taking the equations, $\dfrac{1}{p^2} = A^2x^2 + B^2y^2 + C^2z^2 = \dfrac{G^2}{2T}$ and $Ax^2 + By^2 + Cz^2 = 1$, and eliminating $y$, we have as the projection of the polhode on the $AC$ plane,

$$A\ (B - A)\ x^2 - C\ (C - B)\ z^2 = B - \frac{G^2}{2T}.$$

These projections are therefore hyperbolas. In like manner we see that the projections upon the $AB$ and $BC$ planes are all ellipses.

The case is peculiar for $G^2 = 2BT$, or $p = b$. All the polhodes are projected upon the $AC$ plane as hyperbolas, but in this case the hyperbolas reduce to two straight lines intersecting in the centre. The polhodes in this case become two ellipses and they are called the separating polhodes, because all the polhodes on one side of them enclose the major axis, while all the polhodes on the other side enclose the minor axis. In Fig. 14 these polhodes and various other polhodes are shown.

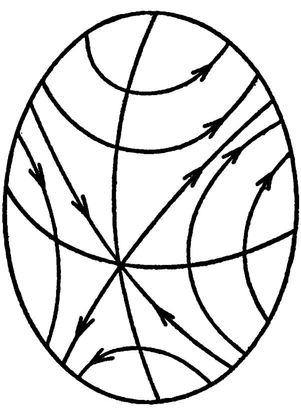

FIG. 14.

Analyzing the motion for this special case we get the following results: From the fundamental equations (3) and (4),

$$\omega_1{}^2 = \frac{C - B}{C - A}\cdot\frac{G^2 - B^2\omega_2{}^2}{AB}$$

$$\omega_3{}^2 = \frac{B - A}{C - A}\cdot\frac{G^2 - B^2\omega_2{}^2}{BC},\ \text{and}\ \dot{\omega}_2 = \frac{C - A}{B}\,\omega_1\omega_3.$$

Putting $\sqrt{\dfrac{(B-A)\,(C-B)}{AC}} = k$, $\dfrac{d\omega_2}{\dfrac{G^2}{B^2} - \omega_2{}^2} = \pm k dt.$

Integrating, $\dfrac{G + B\omega_2}{G - B\omega_2} = Ee^{\pm \frac{2kGt}{B}},$

where $E$ is a determinable constant.

Since $G \cos\beta = B\omega_2$, this becomes

$$\frac{1 + \cos\beta}{1 - \cos\beta} = ctn^2 \frac{\beta}{2} = Ee^{\pm \frac{2kGt}{B}} \qquad (9).$$

From (7) and (8), the invariable and instantaneous cones become in this case two planes intersecting in the middle axis and making angles with the $AB$ plane whose tangents are respectively

$$\pm \sqrt{\frac{C}{A} \cdot \frac{B-A}{C-B}} \text{ and } \pm \sqrt{\frac{A}{C} \cdot \frac{B-A}{C-B}}.$$

In Fig. 15, $B$, $p$, and $I$ are the points respectively where the middle axis, the invariable line and the instantaneous axis pierce a unit sphere about the centre. As the invariable line describes its plane in the body, the arc $Ip$ must always be $\perp$ to the arc $Bp$, and the angle $B$ between the invariable and instantaneous planes is constant. The right spherical triangle $B\,p\,I$ therefore always remains similar, and the problem is reduced to determining the cones described in space by the corners, $B$ and $I$, of this triangle as it rotates about the invariable line with constant angular velocity, $\frac{G}{B}$. From (9) it is evident that as the time increases the angle $\beta$ approaches the value zero or $\pi$. The middle axis therefore

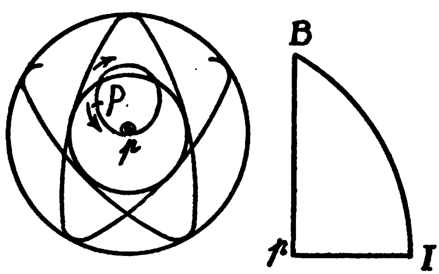

FIG. 15.

moves so as to place itself in coincidence with the invariable line, the direction of the motion being such that like poles, or like rotations, coalesce. The triangle $B\ p\ I$, always remaining similar, finally is reduced to nothing and the motion becomes a steady one about the middle axis.

Differentiating (9), it is seen that the linear velocity along a meridian, $\dot{\beta}$, $p$ being the pole, is proportional to $\sin \beta$, and the linear velocity $\perp$ to this, or along a parallel of latitude, is likewise proportional to $\sin \beta$, since the angular velocity about $p$ is constant. Hence the middle and instantaneous axes, as they spiral inward towards $p$, cut every meridian at a constant angle, and the paths are what are called rhumb lines.

The polhode as it rolls on the invariable plane traces out on this plane a curve called the Herpolhode. Its general character can be seen from Fig. 15. It is limited by two circles which it alternately touches and it is symmetrical about points of tangency. For the special case where the instantaneous axis is in the separating polhode, the herpolhode is quite different. It is shown by the oval in Fig. 15. If the directions of rotation about the middle axis and $p$ are similar, the path curves sharply towards $p$ where at an infinitely small distance it continues to approach $p$ indefinitely. If the rotations about the middle axis and $p$ are in a contrary sense, the instantaneous axis moves in the herpolhode at first away from $p$ and then, the body turning over, curves around sharply from the other direction towards $p$, where as before, from an infinitely small distance, it approaches the pole indefinitely. The instantaneous axis moves to the pole practically in a little less, or a little more, than a quarter of a turn, but since it cannot move directly to the pole and then stop abruptly, it moves first to an infinitely small distance from the pole and then continues its approach indefinitely.

## 29. Gyroscopes under External Forces

Any rotating mass is by definition a Gyroscope. Let us suppose a top, Fig. 16, rotating about its axis with angular velocity $\omega$ and held at an angle $\vartheta_o$ to the vertical. It is then abandoned to gravity and we wish to find the motion. The centre of inertia, $G$, is at a distance, $h$, from the point of support. The moment of inertia about its axis is $C$ and that about a $\perp$ axis through $O$, $A$. We shall take as axes of reference, $OG$, an axis $\perp$ to this in the vertical plane through $O$, and a horizontal axis through $O$ $\perp$ to the other two. If $\dot\psi$ is the angular velocity of the plane $GOV$ about $OV$, the top turns about the $\dot\psi \sin \vartheta$ axis with angular velocity $\dot\psi \sin \vartheta$ and this axis turns about the $OG$ axis with angular velocity $\dot\psi \cos \vartheta$. We have then as the equations of motion

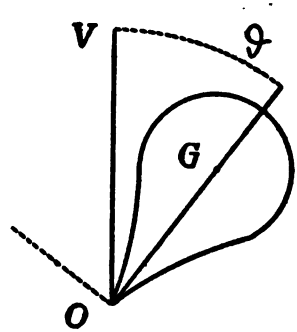


FIG. 16.

$$mgh \sin \vartheta - C\omega\dot\psi \sin \vartheta + A\dot\psi^2 \sin \vartheta \cos \vartheta = A\ddot\vartheta \quad (1)$$

$$C\omega\dot\vartheta - A\dot\psi \cos \vartheta\dot\vartheta = AD_t(\dot\psi \sin \vartheta). \quad (2)$$

Integrating (2), $C\omega(\cos \vartheta_o - \cos \vartheta) = A\dot\psi \sin^2 \vartheta$. (3)

Multiplying (1) by $\dot\vartheta$, (2) by $\dot\psi \sin \vartheta$, adding and integrating

$$mgh(\cos \vartheta_o - \cos \vartheta) = \frac{A\dot\vartheta^2}{2} + \frac{A\dot\psi^2 \sin^2 \vartheta}{2}. \quad (4)$$

Equa. (3) merely states that the moment of momentum about $OV$ remains constant, while Equa. (4) states that the increase of the kinetic energy is equal to the work done — both of which were *a priori* evident. These equations determine the motion completely.

Let us now find the path (guided if necessary) which the axis of the top must take in order that it may move from some point 1 in its actual path to any other point 2 in its actual path, in the shortest time possible.

Since $\sqrt{2gh(\cos\vartheta_o - \cos\vartheta)} = \frac{ds}{dt}$, where $ds$ is an element of path described by the extremity of the radius of gyration $k$, where $mk^2 = A$,

$$dt = \frac{ds}{\sqrt{2gh(\cos\vartheta_o - \cos\vartheta)}},$$

and
$$T = \int_1^2 \frac{ds}{\sqrt{2gh(\cos\vartheta_o - \cos\vartheta)}}.$$

Calling the angle which this path makes with a meridian at any point, $\tau$, $ds = kd\vartheta \sec\tau$. We shall take $\vartheta$ as the independent variable and vary $s$. $\delta ds = kd\vartheta \sec\tau \tan\tau\delta\tau$. Since $\frac{\sin\vartheta d\psi}{d\vartheta} = \tan\tau$, $\delta\tan\tau = \sec^2\tau\delta\tau = \frac{\sin\vartheta}{d\vartheta}\delta d\psi$.

$$\delta\tau = \frac{\sin\vartheta}{d\vartheta}\frac{\delta d\psi}{\sec^2\tau}.\ \text{Hence}\ \delta ds = k\sin\vartheta\sin\tau d\delta\psi$$

and
$$\delta T = \int_1^2 \frac{k\sin\vartheta\sin\tau d\delta\psi}{\sqrt{2gh(\cos\vartheta_o - \cos\vartheta)}}.$$

Integrating, $\delta T = \left.\frac{k\sin\vartheta\sin\tau\,\delta\psi}{\sqrt{2gh(\cos\vartheta_o - \cos\vartheta)}}\right|_1^2 -$

$$\int_1^2 \delta\psi d\cdot\left(\frac{k\sin\vartheta\sin\tau}{\sqrt{2gh(\cos\vartheta_o - \cos\vartheta)}}\right).$$

Since the limits are fixed the first term vanishes and the condition that the path shall be the one of quickest motion is

$$\frac{k\sin\vartheta\sin\tau}{\sqrt{2gh(\cos\vartheta_o - \cos\vartheta)}} = K,$$

where $K$ is an arbitrary constant — say $\frac{C\omega}{2mgh}$. Hence

$$\sin\vartheta\sin\tau = \frac{C\omega}{\sqrt{A}}\sqrt{\frac{\cos\vartheta_o - \cos\vartheta}{2mgh}} \qquad (5)$$

is the equation of the curve in terms of the co-ordinates $\vartheta$ and $\tau$. Substituting for $\sin\tau$,

$$\frac{k\sin\vartheta d\psi}{ds} = \frac{k\sin\vartheta\dot\psi}{\sqrt{2gh(\cos\vartheta_o - \cos\vartheta)}},$$

we have $A\sin^2\vartheta\dot\psi = C\omega(\cos\vartheta_o - \cos\vartheta)$. This identifies the curve with the actual gyroscopic path. Hence the axis of the top moves naturally from one point to another of its

path in the quickest possible time, and generally on a spherical surface a body under the influence of gravity moves from one point to another in the least possible time if it takes a gyroscopic path. If the spherical surface becomes infinitely large these gyroscopic paths become plane cycloids, for $\sin\vartheta$ becomes constant and the meridians become parallel vertical lines, while $\cos\vartheta_o - \cos\vartheta$ measures $y$ from $\cos\vartheta_o$. The rectangular equations of a cycloid are $x = a(\vartheta - \sin\vartheta)$, $y = a(1 - \cos\vartheta)$.

$$\text{Sin}\,\tau = \frac{dx}{\sqrt{dx^2 + dy^2}} = \sqrt{\frac{y}{2a}},$$

or $2a\sin^2\tau = y$ is the equation of a plane cycloid in terms of $\tau$ and $y$.

We have already used this equation in Art. 4, where we found that the cycloid is the curve of quickest passage for a plane surface. We have now found that the gyroscopic path is the curve of quickest passage for a spherical surface.

Taking the equation of the gyroscopic path,

$$\sin\vartheta\sin\tau = \frac{C\omega}{\sqrt{A}}\sqrt{\frac{\cos\vartheta_o - \cos\vartheta}{2mgl}},$$

since at the beginning $\sin\tau = 0$, the path is at first vertically downward. When $\sin\tau = 1$ the path is horizontal and this marks the maximum fall. We have for this point

$$\cos\vartheta = \frac{(C\omega)^2}{4mghA} - \sqrt{1 - \frac{(C\omega)^2\cos\vartheta_o}{2mghA} + \frac{(C\omega)^4}{16m^2g^2h^2A^2}}.$$

There is another value with + between the terms, but this is inadmissible since it makes the value of $\cos\vartheta$ greater than unity. It will be noted that at this point the gyroscopic couple tending to raise the top is just twice the gravitational couple. After this the axis rises symmetrically to its original height where it is momentarily at rest and then repeats the process indefinitely. The path is like a series of festoons hung upon a parallel of latitude at equidistant points. If we suspend the gyroscope so that it can move freely below the point of suspension we have a gyroscopic pendulum. The axis executes a festoon motion,

but will never be in the nadir as long as it has the least rotation. When the rotation ceases there is only a single festoon hung at points 180° apart and this passes through the nadir. From (4), we have in this case

$$t = \int \frac{\sqrt{A}\, d\vartheta}{\sqrt{2mgh\,(\cos\vartheta_0 - \cos\vartheta)}},$$

which, as we have seen, is the law of the ordinary pendulum.

The motion about the vertical axis is the precession, while the motion along a meridian is the nutation. When $\omega$ is large, the amplitude of the vertical vibrations is very small and the vibrations are very rapid, so that in high spinning gyroscopes (tops) the eye cannot detect them, or at most only a slight blurring. But the ear can hear these vibrations and that is the cause of humming in tops. The rapidity of the vibrations can be measured by the note.

With a slight nutation the festoons become very nearly minute cycloids, for making these small relatively to the surface is the same as making the surface very large relatively to the festoons, and in either case these become plane cycloids.

Otherwise, since when $\psi$ and $\vartheta$ become very small we can neglect their squares and products, the equations of motion become

$$mgh \sin\vartheta - C\omega\dot{\psi}\sin\vartheta = A\ddot{\vartheta}$$

$$C\omega\dot{\vartheta} = ADt\,(\dot{\psi}\sin\vartheta).$$

Taking rectangular co-ordinates at the point of rest,

$$mgh \sin\vartheta - C\omega\dot{x} = A\ddot{y}$$

$$C\omega\dot{y} = A\ddot{x}.$$

Integrating

$$x = \frac{mgh \sin\vartheta\, A}{C^2\omega^2}\left[\frac{C\omega t}{A} - \sin\left(\frac{C\omega t}{A}\right)\right]$$

$$y = \frac{mgh \sin\vartheta\, A}{C^2\omega^2}\left[1 - \cos\left(\frac{C\omega t}{A}\right)\right].$$

These are the equations of a cycloid generated by a circle of radius $\frac{mgl \sin\vartheta A}{C^2\omega^2}$, rolling with uniform angular velocity, $\frac{C\omega}{A}$, below a parallel of latitude. The time of a

complete precession, or the time of a complete circuit about the vertical axis, is $\frac{2\pi C\omega}{mgh}$, so that when the rotational velocity is high the precessional velocity is very slow.

If the peg of the top is rounded and the surface upon which it rests rough, so that there is no slipping, the conditions are different. If there is a vertical axis (Fig. 17) through a point $P$ in the axis about which the natural precession is the same as the forced precession due to the rolling of a small circle, $c$, of the peg on the surface, and if the top be given an impulsive velocity about this axis such that the gyroscopic couple exactly balances the gravitational couple about the point $P$, then the motion will be stable. But in ordinary motion with nutations, the peg would, with each dip, roll on a larger circle, thus increasing the forced precession, and the top would rise. With high rotational velocities the natural precession would be very slow while the forced precession would be rapid and consequently the top would rise rapidly. Thus while a top with a peg ending in a mathematical point cannot rise above the level from which it is let go, a top with a rounded peg on a rough surface will rise, and the rise is at the expense of the rotational energy. Brennan has applied this principle of forced precession to balancing a car upon a single rail (v. "The Gyroscope").

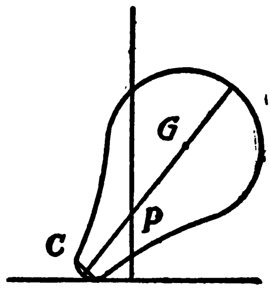

FIG. 17.

### 30. Drift of Rifled Projectiles

Fig. 18 shows a rifled projectile viewed in the line of flight from in front, its long axis making an angle $\vartheta$ with the path of the centre of inertia, $O$. The couple due to the resistance of the air tending to restore the axis to the line of flight increases rapidly up to a certain point with the angle $\vartheta$. The axis and the path at first coincide, but as the

trajectory deviates downward from its original direction, the axis fails to follow it and a couple due to the air resistance strives to bring it in line again. If from any point the path could continue as a straight line, the axis would perform a regular precession with its nutations about this line, as indicated in Fig. 18. It would preserve a constant average inclination, $\vartheta$, to the path, and if the restoring couple be $H \sin \vartheta$ and the moment of inertia about the long axis, $C$, it is seen from Art. 29 that the precessional velocity is $\frac{H \sin \vartheta}{C\omega}$ and the time for a complete precession $\frac{2\pi C\omega}{H}$. But the line about which the long axis strives to describe a regular cone is constantly moving downward. When the point of the projectile has reached its lowest point, $P'$, the line of the path will be at some point, $O'$, below $O$, and if the path should become straight from this point, the precessional cone would become narrower. It is by such an action that generally the long axis is kept close to the line of flight. As the line of flight moves downward the angle $\vartheta$ is always greater in the upper part than in the lower part of the precessional cone. This has an important frictional result. It will be noted that the rotational and precessional motions are such that the outer (away from $O$) surface of the projectile in a certain sense rolls on the air and consequently there is little air-friction on this surface. But the inner surface (towards $O$) moves against the air with both its rotational and precessional velocities. The air is thus a *smooth* surface for the outer surface of the projectile, but *rough* for its inner surface. Since the precessional velocity is much greater in the upper half of the cone than in the lower half, there is a differential effect, the friction

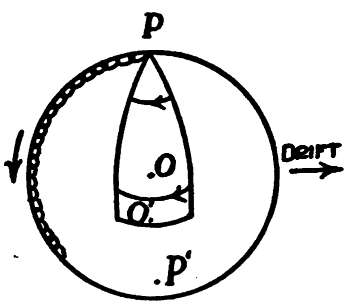

FIG. 18.

on the inner surface in the upper half greatly preponderating. The effect is the same as if the projectile were laid on a partially rough surface and partly slipped and partly rolled parallel to its long axis. If when viewed from behind the projectile is rotating to the left — positive rotation — the precessional motion will be to the right, and the "drift," or the horizontal rolling on the dense underlying cushion of air will be to the left. The axis rolls to the left or the right according to the rifling but always remains parallel to its original vertical plane.

If the downward velocity of the path is equal to, or not greater than, the average downward velocity of the axis in its precession, they will keep together, nearly meeting at the point $P'$. But if the downward velocity of the path is greater than this limit, so that when $P$ reaches $P'$ the path is below this point, then we have the beginning of a "tumble."

The precessional velocity is proportional to $H \sin \vartheta$ and inversely proportional to $\omega$. There is consequently a certain limit, readily calculable, beyond which the axis cannot keep up with a too rapid downward curve of the path. In high angle (mortar) firing, at the vertex of the trajectory the curve is very sharp and at a certain limit, depending upon $H$, $\omega$ and $\dot{\tau}$ where $\tau$ is the angle the path makes with the horizontal, the projectile will tumble. This limit will be reached sooner the greater the value of $\omega$, and therefore when high angles are used a great amount of rifling is not desirable.

There have been many misconceptions as to rotary motion. One is that a gyroscope is a "gyrostat," or device which preserves its plane of rotation. If, by using a mathematical fiction, we could conceive a body spinning with an infinite velocity, then no finite couple could change the direction of its axis and it would be a "gyrostat." But the plane of any finitely spinning gyroscope is readily changed by any couple, albeit the rate of change diminishes as the rotation increases. Another misconception is that the rifling of projectiles is for the purpose of keeping them

"end on." The chief advantage, however, of rifling is to keep the projectile in the gun long enough to have the slow burning powder develop its maximum gas pressure and thus launch the projectile with a velocity otherwise impossible. A spear, any long fusiform body, even an untipped arrow, naturally keeps end on. They are, of course, "stiffer," or have less tendency to slew, the greater the velocity, but any slewing tends to be corrected by the air couple. For high angle firing, therefore, if it were possible to delay the departure of the projectile until the full pressure had been developed, without rifling, a long unrifled projectile would be preferable to a rifled one, since it would not tumble, it would not wobble about the line of flight as a rifled projectile does, and it would not "drift," thus dispensing with an otherwise necessary correction.

From the foregoing it is evident that the axis of a rifled projectile describes a path such as that shown in Fig. 19.

The main curve loops downward and on this are superposed roulettes (cycloids) due to the nutation—ripples as it were on the principal waves. The reverberation of a shell as it passes, in which beats are distinctly audible, is due to these peculiar vibrations of the axis as it executes the major loops.

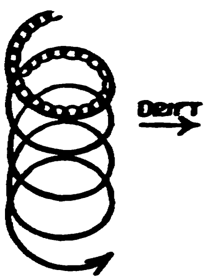


FIG. 19.

## 31. Kepler's Laws

If a body is revolving about another, supposed fixed, and we consider both bodies simply as material points, then it must remain in a fixed plane passing through the two bodies, and its moment of momentum about the central point must remain constant. For there is no couple which could change the plane or moment of momentum of the revolving body.

If it describes an ellipse about the attracting point and this point is a focus of the ellipse, the attraction must vary

inversely as the square of the distance. For, let $M$ be the mass of the attracting body and $m$ that of the revolving body. Writing the equation of the ellipse,

$$r = \frac{a\,(1 - e^2)}{1 + e \cos \varphi} \qquad (1),$$

where $a$ is the major semi-axis, $e$ the eccentricity, and $\varphi$ the angle any radius makes with the minimum radius, and designating the constant moment of momentum, $m\, r^2\, \dot{\varphi}$, by $N$, we have from (1)

$$m\dot{r} = \frac{Ne \sin \varphi}{a\,(1 - e^2)}, \text{ and } m\ddot{r} = \frac{Ne}{a(1 - e^2)} \cos \varphi\dot{\varphi}.$$

The radial force is the difference between the centrifugal and attractional forces, or $m\ddot{r} = mr\dot{\varphi}^2 - mf$, where $f$ is the attractional acceleration. Hence

$$mf = N\dot{\varphi}\left(\frac{1}{r} - \frac{e \cos \varphi}{a\,(-e^3)}\right) = \frac{N\dot{\varphi}}{a\,(1 - e^2)} = \frac{N^2}{mr^2a\,(1 - e^2)} =$$

$$\frac{4\pi^2a^3m}{r^2T^2} = \frac{Mm}{r^2}. \text{ The period, } T = \frac{2\pi abm}{N} = \frac{2\pi a^2\sqrt{1 - e^2}.m}{N},$$

and

$$\frac{T^2}{a^3} = \frac{4\pi^2}{M}. \qquad (2)$$

The attractional force is therefore inversely as the square of the distance, and the ratio of the cubes of the major semi-axes to the squares of the periods is the same for all bodies revolving about the same central fixed body.

This is otherwise evident when we consider that the average centrifugal force must equal the average attractional force, the average of both these forces being that at the average distance. If $R$ is the average distance and $n$ the average angular velocity,

$$n = \frac{2\pi}{T}, \text{ and } \frac{T^2}{R^3} = \frac{4\pi^2}{M} \qquad (3),$$

where $M$ is the mass of the attracting body. This shows by (2) that the average distance of the curve from a focus is $a$. Otherwise, as the sum of the distances of any point from the two foci is always $2a$, the average distance of all the points from one of the foci must, by symmetry, be $a$. The above results are Kepler's Laws.

Kepler's laws, however, are only approximations, since there is, of course, no such thing in Nature as a mass concentrated into a point, or a fixed body. When one of the bodies has a very much greater mass than the other and the distance between them is very great, as is the case for the sun and a planet, these laws are very close approximations. Even so, each body describes an ellipse which has the common centre of inertia of the two bodies for a focus. Considering two bodies of masses $M$ and $m$ to be centrobaric, or homogeneously symmetrical about their centres, the ratio of their accelerations is inversely as their masses. If, therefore we apply to both bodies the acceleration of one of the bodies, reversed in direction, one of them will be brought to rest, while the motion of the other, *relatively* to it, will be unchanged. The acceleration of one body relatively to the other is the sum of their accelerations, and the ratio of the relative acceleration of $m$ to its actual acceleration is $\frac{M + m}{M}$. If then we fix $M$ and increase its mass by $m$, the relative acceleration of $m$ will be unchanged, and its motion relatively to $M$ will be the same as if it revolved about a fixed central body having a mass equal to the sum of the masses. We must therefore write instead of (3), $\frac{T^2}{R^3} = \frac{4\pi^2}{M + m}$, or the ratio of the cubes of the major semi-axes to the squares of the periods of the planets is proportional to the sum of the masses of the sun and each planet, and not simply proportional to the mass of the sun, as Kepler's third law states.

If both bodies are biaxial, we shall see later that this will result eventually in their revolving about each other in a plane which contains both equators and each orbit will be a perfect circle about the common centre of inertia for a centre. Both before and after the merging of their equatorial planes Kepler's third law would not be strictly accurate.

### 32. Attractional Harmonics

In Art. 24 we found that the value of the potential function between two spheroids, homogeneous or made up of homogeneous confocal shells, was

$$V = \frac{MM'}{R} + M'F\left(\frac{I}{R}\right) + MF\left(\frac{I'}{R}\right),$$

where $F\ (\ )$ is a series with ascending powers of $I$ and $R$ respectively in the numerators and denominators. Let us suppose that the bodies are revolving about their common centre of inertia in circular orbits under their mutual gravitation. In investigating the motion of $M$ relatively to $M'$, which we shall consider a material point, we can suppose $M'$ to be fixed and to have a mass $M + M'$. If $\gamma$ be the declination of $M'$, then the couple tending to change $\gamma$, or the rate at which the potential energy is used up about the axis of $\gamma$ is $\frac{dV}{d\gamma}$, and this gravitational couple will be of the general form

$$G = \frac{dV}{d\gamma} = -a\frac{dI}{d\gamma} + bI\frac{dI}{d\gamma} - cI^2\frac{dI}{d\gamma} + \text{etc.},$$

where $a$, $b$, $c$, etc., are determinable constants. Since $I = C\cos^2\gamma + A\sin^2\gamma$ the gravitational couple will have the general form

$$G = (a - bI + cI^2 - dI^3 + \text{etc.})\sin\gamma\cos\gamma \qquad (1),$$

where $a$, $b$, $c$, are other constants. It will be noted that the couple vanishes when $R$ passes through a principal axis and therefore altogether when the body is uniaxial.

We shall call the plane containing the two equal axes the equatorial plane, and it will be convenient to resolve the gravitational couple into a couple about the line where the equatorial and orbital planes intersect — the nodal line — and a couple about an axis $\perp$ to this in the equatorial plane. If $\dot\psi$ is the precessional velocity of the $C$ axis about the orbital pole, we may call the first axis the $\dot\vartheta$ axis and the second axis the $\dot\psi\sin\vartheta$ axis.

For the purposes of our problem it is indifferent whether we consider $M'$ fixed while $M$ revolves about it, or $M$ fixed and $M'$ revolving in the same orbit. Let $NAB$ be the orbit of the attracting body, $E$ its pole, $O$ the centre of $M$, and $ON$ the nodal line. We shall suppose the orbit to be a perfect circle. Let the attracting body be at $A$ and moving with constant angular velocity, $a$. Taking our measures from $N$, since $\psi$ is small compared with $a$, we can for a short time consider the angle $NOA$ as sensibly equal to $at$. $\gamma = POA$, and the gravitational couple is in this plane. Let the angle between the planes $AOP$ and $BOP$ be $\varphi$. Then the $\vartheta$ component is $G \cos \varphi$ and the $\psi \sin \vartheta$ component is $G \sin \varphi$. By spherical trigonometry,

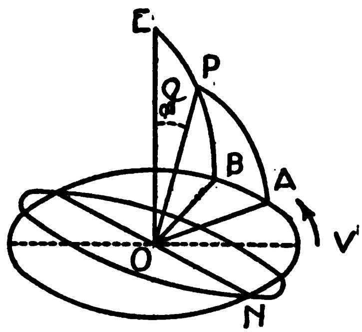

FIG. 20.

$$\cos \varphi = \frac{\cos \vartheta}{\sqrt{\cos^2 \vartheta + ctn^2 at}} \text{ and } \sin \varphi = \frac{ctn\, at}{\sqrt{\cos^2 \vartheta + ctn^2 at}}.$$

$$\cos \gamma = \sin \vartheta \sin at \text{ and } \sin \gamma = \sqrt{1 - \sin^2 \vartheta \sin^2 at}.$$

Since $$\sin at = \sqrt{\frac{1 - \sin^2 \vartheta \sin^2 at}{\cos^2 \vartheta + ctn^2 at}},$$

the $\vartheta$ component of the term $kI^n \sin \gamma \cos \gamma$ in (1) becomes $kI^n \sin \vartheta \cos \vartheta \sin^2 at$, and the $\psi \sin \vartheta$ component becomes $kI^n \sin \vartheta \sin at \cos at$.

Since the nutation is small compared with $\vartheta$, we may consider $\vartheta$ as sensibly constant, and since

$$I = A + (C - A) \sin^2 \vartheta \sin^2 at,$$

the total $\vartheta$ component can be thrown into the general form

$$A\ddot{\vartheta} = -b_1 \sin^2 at + b_2 \sin^4 at - b_3 \sin^6 at \ldots\ldots \pm b_n \sin^{2n} at + \text{etc.} \qquad (2).$$

Likewise the $\psi \sin \vartheta$ component can be thrown into the general form

$$A\psi \sin\vartheta = c_1 \sin at \cos at - c_2 \sin^3 at \cos at + c_3 \sin^5 at \cos at \; . \; . \; . \; . \pm c_n \sin^{2n-1} at \cos at + \text{etc.} \quad (3).$$

$\sin^{2n} at$ can be written as the limited series,

$$\sin^{2n} at = k - k_1 \cos 2\,at + k_2 \cos 4\,at - k_3 \cos 6\,at \; . \; . \; . \; . \pm k_n \cos 2n\,at \quad (4).$$

For

$$\cos 2\,at = 1 - 2\sin^2 at$$
$$\cos 4\,at = 1 - 8\sin^2 at + 8\sin^4 at$$

and so on.

Conversely, $\cos 2n\,at$ can be written as the limited series,

$$\cos 2n\,at = m - m_1 \sin^2 at + m_2 \sin^4 at - m_3 \sin^6 at \; . \; . \; . \; . \; . \pm m_n \sin^{2n} at. \quad (5).$$

Hence the $\vartheta$ couple has the general form

$$A\vartheta = b - b_1 \cos 2\,at + b_2 \cos 4\,at - b_3 \cos 6\,at \; . \; . \; . \; . \; . \pm b_n \cos 2n\,at. \quad (6).$$

By differentiating (4) we have the series

$$\sin^{2n-1} at \cos at = c_1 \sin 2\,at - c_2 \sin 4\,at + c_3 \sin 6\,at \; . \; . \; . \; . \pm c_n \sin 2n\,at \quad (7),$$

and we can throw the $\psi \sin\vartheta$ couple into the general form of (7).

Combining any two terms of the series (6) and (7) having the same period, it is evident that we have an elliptic harmonic motion, the axes of the ellipse being determinable in each case. The fundamental period is $\frac{\pi}{a}$, $a$ being the angular velocity of the attracting body, and this is the period of the first ellipse due to the two couples, $-b_1 \cos 2\,at$ and $c_1 \sin 2\,at$. The following ellipses represent the higher harmonics of this fundamental period, viz.,

$$\frac{\pi}{2a}, \frac{\pi}{3a}, \frac{\pi}{4a}, \frac{\pi}{5a}, \text{ etc.}$$

Fig. 21 represents the first four ellipses viewed from outside the orbit. The long arrow shows the direction of motion of the attracting body. The first ellipse is vertical, the second horizontal and so on alternately, the motion in the vertical ellipses being always to the left, while that in the horizontal ellipses is to the right.

Taking the case of the earth and the moon, since the

distance between these bodies is great, the gravitational series (1) decreases very rapidly and the first ellipse is the only one which is appreciable. Let us investigate this ellipse. Taking the first term of

$$(M + M^1)\, F\left(\frac{I}{R}\right),$$

we readily find that $A\ddot{\vartheta} = -K \sin\vartheta \cos\vartheta \sin^2 at$ (8)

and $A\ddot{\psi} \sin\vartheta = K \sin\vartheta \sin at \cos at$ (9),

$$\text{where } K = \frac{3\,(M + M^1)}{R^3}\,(C - A).$$

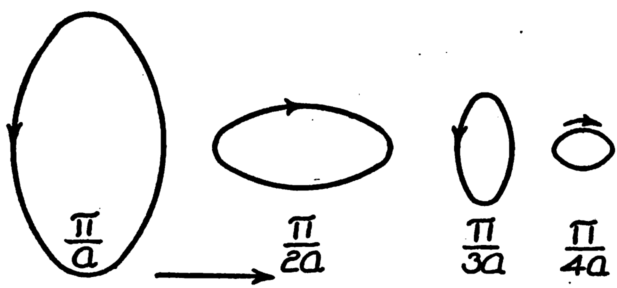


FIG. 21.

The elliptic motion due to these two couples can be represented by a material point, or particle, moving harmonically in a vertical ellipse, to the left, with constant angular velocity $2a$. The major axis of the ellipse is $\frac{K}{2C\omega} \sin\vartheta$, and the minor axis is $\frac{K}{2C\omega} \sin\vartheta \cos\vartheta$. The vertical velocity in the ellipse will be $\dot{\vartheta} = -\frac{K}{2C\omega} \sin\vartheta \sin 2\,at$ and the horizontal velocity, $\dot{\psi} \sin\vartheta = \frac{K}{2C\omega} \sin\vartheta \cos\vartheta \cos 2\,at$. If now we suppose the ellipse to move bodily to the left with a horizontal angular velocity $-\frac{K}{2C\omega} \sin\vartheta \cos\vartheta$, the total horizontal velocity of the particle will be $\underline{\dot{\psi} \sin\vartheta} =$

$$\frac{K}{2C\omega} \sin \vartheta \cos \vartheta \, (\cos 2\,at - 1) = -\frac{K}{C\omega} \sin \vartheta \cos \vartheta \sin^2 at.$$

Substituting the extremity of the axis of the earth for the particle, the horizontal angular velocity $\dot{\psi} \sin \vartheta$ will gyroscopically cause a vertical angular acceleration,

$$K \sin \vartheta \cos \vartheta \sin^2 at,$$

and the vertical velocity, $\dot{\vartheta} = -\frac{K}{2C\omega} \sin \vartheta \sin 2\,at$, will cause a horizontal angular acceleration $-\frac{K}{2C\omega} \sin \vartheta \sin 2\,at$. But these gyroscopic couples are exactly equal and opposite to the gravitational couples (8) and (9). We have seen that gyroscopic couples are purely internal forces, representing merely the moments of the centrifugal forces, which are forces of inertia. The gravitational couples are the applied forces and the gyroscopic couples are the forces of inertia, due to the motion. These forces are exactly balanced and therefore the axis of the earth moves freely (without constraint) in the first ellipse. The axis of the earth executes a harmonic motion in the first ellipse with constant angular velocity, $2a$, in a counter clockwise direction, while the ellipse itself performs a constant horizontal retrograde precession about the pole of the moon's orbit with angular velocity, $\dot{\psi}_o = -\frac{K}{2C\omega} \cos \vartheta_o$, where $\vartheta_o$ is the constant inclination of the centre of the ellipse.

In Equa. (6) there is a single unpaired term, $b$. This is equal to $\dot{\psi}_o \sin \vartheta_o = -\frac{K}{2C\omega} \sin \vartheta_o \cos \vartheta_o$. The motion of the mean position of the axis, $\vartheta_o$, thus produces a gyroscopic couple, $C\omega\dot{\psi}_o \sin \vartheta_o$, which exactly balances the gravitational couple for this inclination, viz., $-\frac{K}{2} \sin \vartheta_o \cos \vartheta_o$, with a resulting constant and smooth precession of this mean position, viz., the centre of the harmonic ellipse.

It will be noted that in deriving Equa. (6) every term

gave a harmonic not only of the order of the term but also of all the lower harmonics, together with a constant (zero harmonic) which is a part of $b$, the couple producing the constant retrograde precession. The constant precession is therefore represented by a series made up of alternating plus and minus terms, and the value derived from the first ellipse alone, viz., $\dot{\psi}_o = -\frac{K}{2C\omega}\cos\vartheta_o$, is slightly greater than the actual value. The actual motion is the following: The centre of the first ellipse executes a constant retrograde horizontal precession. A point in this ellipse moves with a constant angular velocity, $2a$, in a positive direction (to the left). About this point another point describes the second ellipse with an angular velocity $4a$ in a negative direction. About the last point another point describes the third ellipse with an angular velocity $6a$, and so on, while the axis of the earth moves in the last ellipse of all.

Whenever the couple ceases, the axis comes instantly to rest, and starting from any position it immediately falls into motion in that part of the harmonic ellipse necessary to bring it to rest at the node. Considering only the first ellipse, the inclination of the axis to the pole of the orbit is always greatest at the nodes and least at quadratures. That is, the axis is always at the bottom of the ellipse at the nodes. Where two biaxial bodies revolve about each other at a distance which is not an excessive multiple of their diameters — and there are such instances among heavenly bodies — not only the fundamental period but a number of the higher harmonics would be appreciable, constituting a veritable "Music of the Spheres."

### 33. Simplification of Motion

In investigating the precession of a planet we saw that the gravitational couple tending to bring the equatorial plane into the line joining the centres of inertia of the two bodies was

$$K \sin \gamma \cos \gamma = \frac{3\,(M + M^1)}{R^3}\,(C - A) \sin \gamma, \cos \gamma,$$

where $\gamma$ is the declination of the attracting body. But the action is mutual and an equal couple strives to bring the attracting body into the plane of the equator. This couple causes the plane of the orbit to precess about the pole of the planet just as the attractional couple of the satellite causes the polar axis of the planet to precess about the pole of its orbit. Let us suppose a satellite, Fig. 22, to revolve about a spheroidal planet at a constant distance from $O$. If the planet were a sphere, the satellite would describe a circular orbit $NAN'$. The component of our couple in the direction of the path of the satellite is $K \sin \gamma \cos \gamma \cos \tau$, where $\tau$ is the angle the path makes with a meridian. The velocity, instead of being constant, will therefore be retarded from $N$ to $A$ and accelerated from $A$ to $N'$, but the velocity in the equator will always be the same, viz., the velocity for the circular orbit. It is evident, therefore, that a satellite cannot describe a circular orbit about a biaxial body, unless it keeps in the equatorial plane. It executes a path $NBC$ wholly within the circular orbit, and making the same angle with the equatorial plane at $N$ and $C$ as the uninfluenced path $NAN'$. The node $C$ occurs before the node $N'$ and the effect of the couple is to make the nodes regress, while the average inclination of the path to the plane of the equator and the average velocity remain constant. The inclination of the plane of the path to the plane of the equator is least at the summit, $B$, and greatest at the nodes, where it is the constant inclination of the uninfluenced path. The motion of the orbit is the same as that of a solid ring, into which we may suppose the mass of the satellite to be uniformly distributed,

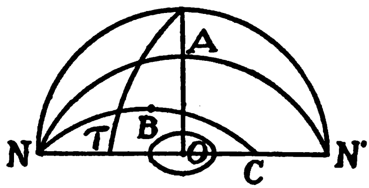


FIG. 22.

rotating with the same angular velocity. A point on this ring gives the position of the satellite at any time. The motion of such a ring is obviously a precession of its axis about the pole of the planet, accompanied by nutations, precisely as in the case of a top.

The above is on the supposition that the satellite maintains a constant distance from the planet. Actually such a condition is only possible when the orbit coincides with the equatorial plane.

Actually the orbit not only precesses, but gradually loses its inclination until it finally coalesces with the equatorial plane, in somewhat the same way as a plate spinning on its edge on a table eventually coincides with the table. A biaxial body eventually brings any revolving body permanently into the plane of its equator. Rigorous proofs of this have been given by Laplace and Tisserand. An informal explanation may be found in "Popular Astronomy," Sept. 1915.* Having got our satellite into the equatorial plane, let us see what happens next. A body launched in the equatorial plane will, in general, describe an orbit which is nearly an ellipse, although not exactly an ellipse, provided the planet is biaxial. Taking first the case of a spherical planet of mass, $M$, the orbit will be an ellipse and by Art. 32 it is readily seen that $\frac{1}{r_1} = \frac{2M}{N^2} - \frac{1}{r_2}$, where $r_1$ is the least distance of the orbit from the planet, $r_2$ the greatest distance, and $N = r^2\dot{\varphi}$. If now we suppose the planet to be slightly flattened, the attraction at all points in the equatorial plane will be increased, so that starting from $r_1$ the corresponding maximum distance $r_2^1$ will be shorter than $r_2$. The path will be nearly an ellipse corresponding to a slightly greater mass at the focus. Likewise starting from $r_2$, the corresponding minimum distance $r_1^1$ will be shorter than $r_1$.

$$\frac{1}{r_1^1} = \frac{2\ (M + dM)}{N^2} - \frac{1}{r_2}, \text{ and } \frac{1}{r_2^1} = \frac{2\ (M + dM)}{N^2} - \frac{1}{r_1}.$$

* Some problems in Gravitational Astronomy.—*The Author.*

Whence $$\frac{r_2 - r_2^1}{r_1 - r_1^1} = \frac{r_2 r_2^1}{r_1 r_1^1} \text{ and } \frac{r_1^1}{r_1} > \frac{r_2^1}{r_2} \quad (1).$$

That is, owing to the flattening of the planet, the satellite describes an approximate ellipse with a major axis which is slightly less than that of the original ellipse and the maximum distance from the focus is decreased by a greater amount than the minimum distance. Calling the eccentricity of the original ellipse, $e$, and that of the new approximate ellipse, $e^1$, we have from (1) $\frac{1 - e^1}{1 + e^1} > \frac{1 - e}{1 + e}$, or $e > e^1$, or the new approximate ellipse is less eccentric than the original one. It will be seen that in the inward journey the new path is within the original ellipse, while on the outward journey it is without. Thus the inward half of the new path is more eccentric than the old path, while the outward half is less eccentric, but on the whole the new path is less eccentric. It is further evident that owing to the greater eccentricity of the inward half, the minimum radius will be slightly ahead of the old one, while owing to the lesser eccentricity of the outward half the maximum radius will be behind the old one.

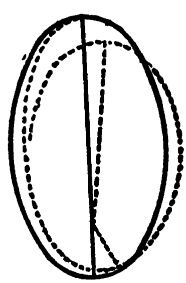

FIG. 23.

In other words, the major axis progresses at minimum distance and regresses at maximum distance, but the former exceeds the latter, so that on the whole the major axis progresses, or moves in the direction of the motion with each revolution. In Fig. 23, the full line represents the original ellipse and the dotted line the transformed path. We see then that a satellite revolving in the equatorial plane of a biaxial planet: 1. Alternately increases and decreases its eccentricity, but on the whole progressively decreases it, until it revolves in a perfect circle. 2. The approximate major axis, by alternate progressions and regressions, on the the whole progresses. 3. The semi-major axis, or mean distance, pro-

gressively decreases until the final circular orbit is attained. Better, however, than any theoretical proof is the direct experimental proof which meets our eyes at many points of the heavens. All the nearer satellites revolve nearly in their equatorial planes in almost perfect circles, and they would perform these motions *exactly* if it were not for the disturbing action of the sun. There can be no more beautiful experiment than the following: Suppose some power able to hurl masses of matter at some planet isolated in space. The planet would catch them, and winding them about itself would gradually bring them all into its equatorial plane, moving in nearly perfect circles. A single satellite would describe an exact circle.

We shall see directly that the disturbing action of the sun causes the orbits of satellites to assume a compromise position between the equatorial and orbital planes of the planet. The plane about the axis of which an orbit performs its precessions is called the fundamental plane of the orbit. It is not necessary that the influencing body should be within the orbit, for a distant body can likewise produce a precession of the orbit. The sun causes the moon's orbit to precess exactly as the earth's equatorial protuberance does, and it happens that the sun's influence is considerable, due to his great mass, while the earth's influence is slight. We may represent such a precession by a vector perpendicular to its fundamental plane, having a length equal to the precessional velocity, and in the case of several influencing bodies we can compound the effect by compounding the vectors. Thus the precession of the moon's orbit due to the sun has the ecliptic for its fundamental plane, while the precession due to the earth has the earth's equatorial plane for its fundamental plane. The resultant precessional axis lies in a plane containing the axis of the ecliptic and the earth's polar axis, and inclined to the former about 12′. The inclination of the moon's orbit to this resultant axis is about 84° 40′. Hence as the moon's orbit rotates about this resultant axis, its in-

clination to the ecliptic varies from a maximum of 5° 20′ to a minimum of 4° 56′.

This effect of the equatorial protuberance of a planet in bringing a satellite into its plane and then destroying its eccentricity, is very strong when the satellite is near the planet. It is strikingly shown in the case of the satellites of Mars and of all the nearer satellites of our system. The equatorial planes of the planets are, of course, constantly shifting, due to planetary precession, but they carry their nearer satellities with them practically the same as if their orbits were rigidly attached.

These gravitational effects all exemplify a general principle in Nature which we may call the Simplification of Motion. There is everywhere a tendency to reduce complicated and irregular forms of motion to simpler and more regular forms. By the development of gyroscopic couples, two or more rotations tend to fuse into a single rotation. This tendency may result only in an oscillation about the position of fusion (equilibrium) but frictional forces eventually effect the fusion. The motion of a triaxial body with its instantaneous axis in the separating polhode is an example of the simplification of motion. Tidal forces tend to equalize rotational and revolutional motions and eventually do equalize them — this being the simplest form of such a double motion. We shall see directly that all rotational planes tend to coalesce with revolutional planes.

The first, second, and third satellites of Jupiter are an example of the harmonizing of motions. Considering a revolution as a vibration, and circular orbits are composed of two simple harmonic motions perpendicular to each other, the frequencies of these vibrations are $\frac{1}{T_1}$, $\frac{1}{T_2}$, and $\frac{1}{T_3}$, where $T$ is the period. By their mutual interactions, they have been able to bring their frequencies into a simple harmonic relation. The frequencies are very nearly as $1, \frac{1}{2}, \frac{1}{4}$. The five inner satellites of Saturn have frequencies

nearly as $\frac{1}{11}\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{6} \ldots \frac{1}{10}\right)$. A vibrating body not only tends to set up harmonic vibrations in other bodies, but when that is impossible and the shape is changeable, actually tends to shake them into forms capable of such harmonics. A rigid body can only respond to certain fixed frequencies, but an elastic body may adjust itself to the proper frequencies.

The mutual tendency of the orbits of our system is to coalesce into a single plane, and, given time enough, they will eventually coalesce into the Invariable Plane of the system. And nowhere is there an opposite tendency. Simple and regular motions never degenerate into complex and irregular forms, for the simpler the motion the more stable does it become, and all irregular and complex forms are essentially unstable.

### 34. Effect of Moon's Orbital Precession on Earth's Axis

Owing chiefly to the sun's action, the moon's orbit performs a complete precession, with the ecliptic as its fundamental plane, in about 18⅔ years. This in turn exerts an influence on the earth's axis which we shall now examine. Let $A$, Fig. 24, be the position of the earth's axis, $C$ the pole of the ecliptic, $\vartheta$ the angle $CA$, $B$ the pole of the moon's orbit, and $CB$ the constant angle, $a$. Actually $\vartheta$ is about 23° 27′ and $a$ is nearly 5°. It is evident that we can effect the steady retrograde precession caused by a revolving body, by fixing half its mass at the pole of its orbit and supposing it to exert a repellent instead of an attractive action. We have then to consider the action of a body having half the moon's

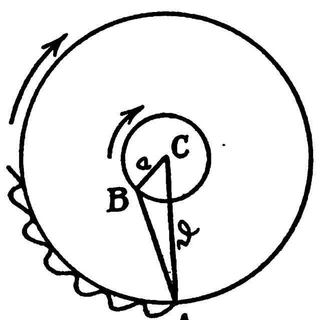

FIG. 24.

mass and moving in a retrograde direction in the small circle with constant angular velocity, $-b$. Let the angle $BA$ be $c$. Then the gravitational couple is $-K \sin c \cos c$. If the angle $CAB = A$, the $\dot{\vartheta}$ and $\dot{\psi} \sin \vartheta$ couples are $-K \sin c \cos c \cos A$ and $K \sin c \cos c \sin A$.

Since $\dot{\psi}$, the precession of the earth's axis about $C$, is small compared with $b$, the angle $ACB = C$, measured from a position of conjunction, will for a short time be sensibly equal to $-bt$. $\dfrac{\sin A}{\sin C} = \dfrac{\sin a}{\sin c}$ (1) and $\cos c = \cos a \cos \vartheta +$

$$\sin a \sin \vartheta \cos C. \qquad (2).$$

Hence the

$$\dot{\vartheta} \text{ couple is } -K \cos c \sqrt{\sin^2 c - \sin^2 a \sin^2 C}.$$

$\sin^2 c - \sin^2 a \sin^2 C = (\cos a \sin \vartheta - \sin a \cos \vartheta \cos C)^2$, and the $\dot{\vartheta}$ couple is

$$-K \cos c (\cos a \sin \vartheta - \sin a \cos \vartheta \cos C).$$

Substituting the value of $\cos c$ from (2) this becomes

$$A\ddot{\vartheta} = -K \cos^2 a \sin \vartheta \cos \vartheta + K \sin a \cos a \cos 2\vartheta \cos C + K \sin^2 a \sin \vartheta \cos \vartheta \cos^2 C. \ (3).$$

Likewise

$$A\dot{\psi} \sin \vartheta = K \sin c \cos c \sin A = K \sin a \cos a \cos \vartheta \sin C + K \sin^2 a \sin \vartheta \sin C \cos C. \qquad (4).$$

Our equations of motion therefore are

$$K \sin^2 a \sin \vartheta \cos \vartheta \cos^2 bt + K \sin a \cos a \cos 2\vartheta \cos bt - K \cos^2 a \sin \vartheta \cos \vartheta - C\omega\dot{\psi} \sin \vartheta + A\dot{\psi}^2 \sin \vartheta \cos \vartheta = A\ddot{\vartheta}. \ (5).$$

$$-K \sin^2 a \sin \vartheta \sin bt \cos bt - K \sin a \cos a \cos \vartheta \sin bt + C\omega\dot{\vartheta} - A\dot{\psi} \cos \vartheta\dot{\vartheta} = A\ddot{\psi} \sin \vartheta \qquad (6).$$

[We have dropped the angles $A$ and $C$ and these symbols now resume their usual significance.]

$\dot{\vartheta}$ and $\dot{\psi}$ are so small that we can neglect second powers and $\vartheta$ is sensibly constant. The motion is so small that we can use $\dot{x}$ in place of $\dot{\psi} \sin \vartheta$ and $\dot{y}$ in place of $\dot{\vartheta}$. In other words we can use rectangular in place of spherical co-ordinates. Hence our equations of motion become,

$$K \sin^2 a \sin \vartheta \cos \vartheta \cos^2 bt + K \sin a \cos a \cos 2\vartheta \cos bt - K \cos^2 a \sin \vartheta \cos \vartheta - C\omega\dot{x} = A\ddot{y} \ (7)$$

and

$$-K \sin^2 a \sin \vartheta \sin bt \cos bt - K \sin a \cos a \cos \vartheta \sin bt + C\omega\dot{y} = A\ddot{x} \qquad (8).$$

Integrating (8), $C\omega y - A\ (\dot{x} - \dot{x}_0) =$

$$K \sin^2 a \sin \vartheta \frac{\sin^2 bt}{2b} -$$

$$K \sin a \cos a \cos \vartheta \frac{(\cos bt - 1)}{b}.$$

Since $\dot{x}$ is small compared with $\omega$, we can write this

$$C\omega y = K \sin^2 a \sin \vartheta \frac{\sin^2 bt}{2b} +$$

$$K \sin a \cos a \cos \vartheta \frac{(1 - \cos bt)}{b} \qquad (9).$$

Integrating (7),

$$C\omega x = K \sin^2 a \sin \vartheta \cos \vartheta \left(\frac{t}{2} + \frac{(\sin bt \cos bt)}{2b}\right) +$$

$$K \sin a \cos a \cos 2\vartheta \frac{\sin bt}{b} - K \cos^2 a \sin \vartheta \cos \vartheta . t \qquad (10).$$

Plotting the curve from these equations we find that it has the shape given in Fig. 24.

Starting with the origin at the time when the poles are on the same celestial meridian (conjunction), the inclination of the earth's axis to the pole of the ecliptic is here a minimum. After this it increases until a maximum is reached with $bt = \pi$. It then regains its former minimum when $bt = 2\pi$ and the two poles are again in conjunction. The path of the earth's axis is thus an unsymmetrical wavy curve. There are about 1400 such complete waves in every complete precessional circle of the earth's axis about the pole of the ecliptic. The variation of $\vartheta$, or the depth of the curve is about 9″.

## 35. Effect of the Moon's Orbital Precession on her Own Axis

This action upon the earth's axis, due to the shifting of the moon's orbit, is purely reciprocal.

The earth's attraction upon the moon's equatorial protuberance causes the moon's axis to precess (retrograde) about the pole of her own orbit. As far as the precessional effect is concerned, it is a matter of indifference whether

the moon revolves about the earth or the earth revolves about the moon in the moon's orbit. We can cause the same precessional effect upon the moon's axis by supposing half the earth's mass to be at the pole of the moon's orbit exerting a repulsional instead of an attractional action. If then in Fig. 24 we suppose $A$ to be the pole of the moon's orbit moving with a constant retrograde precession, $-b$, and half the earth's mass to be at this pole, while the moon's axis is at $B$, the problem, though reversed, is exactly similar to the previous one. The action of the earth will be to cause the moon's axis to precess in the small circle, $B$, with nutations, forming a wavy curve precisely as in the other problem. Calling now the angle $CA$, $\vartheta$, and the angle $CA$, $a$, the equations of motion are

$$K \sin^2 a \sin \vartheta \cos \vartheta \cos^2 (\psi - bt) +$$
$$K \sin a \cos a \cos 2\vartheta \cos (\psi - bt) - K \cos^2 a \sin \vartheta \cos \vartheta -$$
$$C\omega\dot{\psi} \sin \vartheta = A\ddot{\vartheta} \text{ (11) and}$$

$$K \sin^2 a \sin \vartheta \sin (\psi - bt) \cos (\psi - bt) +$$
$$K \sin a \cos a \cos \vartheta \sin (\psi - bt) + C\omega\dot{\vartheta} = A\ddot{\psi} \sin \vartheta \quad (12),$$

where $K$ and $\psi$ now refer to the moon. The angle, $\psi - bt$, is the difference between the precession of the moon's axis and the precession of the pole of her orbit.

Calling $\psi - bt$, $\alpha$, the equations of motion can be written

$$D \cos^2 \alpha + E \cos \alpha - F - C\omega\dot{\psi} \sin \vartheta = A\ddot{\vartheta} \quad (13).$$
$$G \sin \alpha \cos \alpha + H \sin \alpha + C\omega\dot{\vartheta} = A\ddot{\psi} \sin \vartheta \quad (14),$$

where $D$, $E$, $F$, etc., are determined constants. The natural independent precessions of the two poles we are considering, viz., $\psi$ the precession of the moon's axis and $bt$ the precession of the pole of her orbit, are not the same. However, they are forced into coincidence by a peculiar action which we shall now discover. The natural precessions being different, one pole will eventually overtake the other and at some time $\alpha$ will be momentarily zero, and at some other time momentarily $\pi$ — conjunction or opposition. From Fig. 24, we see that in these positions $\dot{\vartheta}$ is momentarily zero.

Let us suppose that $\alpha$ has become $\pi$, the two poles

being at their maximum distance apart, with $C$ the pole of the ecliptic between them. From (14), the precessional acceleration, $\ddot{\psi} \sin \vartheta$, is here zero, and the precessional velocity is momentarily constant. But $\dot{\psi}$ and $b$ not being equal, the moon's axis will directly either get ahead of or lag behind the orbital pole. If the angle $\alpha$ becomes negative, from (14) the precessional acceleration becomes negative and a couple is brought into play tending to bring the moon's axis back into coincidence (opposition) with the orbital pole. If the moon's axis gets ahead and the angle $\alpha$ becomes positive, from (14) a positive couple arises tending to turn the moon's axis back into coincidence with the other pole.

There is a limit beyond which this regulatory couple could not overcome the difference between the natural velocities, but in the case of the moon, her mass being slight, the couple is well within this limit.

When $\alpha$ is zero, if the moon's axis gains from this position, $\alpha$ becomes negative and a negative couple arises which tends to increase the gain still further, while if the moon's axis lags, $\alpha$ becomes positive and a positive couple arises which tends to set it still further back. Hence, when the two poles are in conjunction they exert a mutually repellant action and are in unstable equilibrium, while when they are in opposition they are in stable equilibrium.

We have here another case of the simplification of motion. Instead of pursuing an irregular motion with two independent precessions, the poles fall into step 180° apart, and the motion is afterwards performed as if the moon and her orbit were rigidly connected, moving together as a whole in the symmetrical position where the moon's axis, the axis of her orbit and the axis of the ecliptic all lie in one plane. The moon's axis thus moves with a nearly constant precession and with a nearly constant inclination to the pole of the ecliptic — practically a Poinsot motion, or a motion under the action of no forces.

The same regulatory couple exists for the earth's axis in the problem previously considered, and there is a tendency to force the earth's precession to keep step with the precession of the moon's orbital pole, but the masses being reversed, the regulatory couple is unable to force the earth's axis to the proper velocity and it constantly lags behind. The distortion of the curve in Fig. 24 plainly indicates the effort which the orbital pole makes to carry the earth's axis along with itself.

This peculiar motion of the moon's axis has been exactly confirmed many times by observation. Cassini first discovered it in 1675 by observation, and it is known as Cassini's Theorem. It is usually stated thus: "The plane of the moon's orbit, her equatorial plane, and a plane through her centre parallel to the ecliptic, always intersect in the same line, and the ecliptic plane always lies between the other two."

In all revolving systems, both the central and the satellitic bodies always tend to fall into a Cassini motion, and the smaller bodies generally acquire such a condition at an early stage. It is quite certain that all the nearer satellites of our system execute Cassini motions, and from tidal forces all the nearer, and probably also the remoter ones, perform their rotations and revolutions in the same period. There is however no connection between the two phenomena except in so far as a low rotational velocity favors the action of the regulatory couple.

A certain historical interest attaches to Cassini's theorem. Shortly after this peculiar motion was discovered, it was perceived that there must be some cause and an explanation was eagerly sought. In 1754, D'Alembert attempted a solution without success. Finally in 1764 the French Academy offered a prize for the discovery of the cause and this prize was won by Lagrange in 1780. His solution, however, was only a partial one. Lagrange proved that if the moon is triaxial with the axis of least moment, always pointing nearly towards the earth, then, such a

condition of the three planes once existing, it would persist. Routh, in his "Advanced Dynamics," has given a proof along similar lines. Both of these proofs postulate that the axis of least moment shall always point approximately towards the earth. But we have seen that the coincidence of the rotational and revolutional periods is not essential and the body need not be triaxial. In fact we have supposed the moon to be biaxial.

## 36. Glacial Epochs

We have seen that two bodies revolving about their common centre of inertia are subject to tidal forces tending to tear each body apart from its centre towards and away from the other body. They are thus lengthened in the direction of the line between them and compressed in a direction perpendicular to their orbits. If the rotation and revolution are not the same, the matter of the body in rotating through these body tides is subjected to a kneading process which reduces its rotational velocity by transforming rotational energy into heat. In the case of the earth these body tides are not inappreciable. They are slight but they certainly exist, and the continuous operation of even a slight action for immense periods of time has, as we shall see, far-reaching effects.

The earth is rotating at all times about two axes — the polar axis and the precessional axis which is perpendicular to the former. As these rotations have to be executed through the tidal distortion, they are constantly being opposed. The angular velocity of the tide is the orbital angular velocity, $a$, of the attracting body — sun or moon — and for both bodies the average axis of the tide is perpendicular to the ecliptic. If $H$ is the tidal effect, or couple, we may suppose it decomposed into two tides, $H \cos \vartheta$ about the diurnal axis and $H \sin \vartheta$ about the precessional axis. The effect of either tide is proportional to the difference of the tidal and rotational velocities, or

the couple about the diurnal axis is proportional to $H \cos \vartheta$ $(a - \omega)$, while that about the precessional axis is proportional to $H \sin \vartheta\ (a - \dot{\psi} \sin \vartheta)$. The former couple is negative and tends to reduce $\omega$ to $a$, while the latter is positive and tends to reduce the negative precession, $\dot{\psi} \sin \vartheta$.

The constant negative precessional velocity, $\dot{\psi} \sin \vartheta$, which is just sufficient to balance the average gravitational couple and maintain the inclination constant, we have found to be $-\frac{K}{2C\omega} \sin \vartheta \cos \vartheta$. If we reduce this negative velocity by braking, or accelerate positively, it will no longer be able to support the gravitational couple and the axis will yield in part to this couple. In the case of a top, which is precisely similar, if we reduce the precession by the slightest amount, it begins to fall, and if we abolish the precession altogether it falls exactly as if there were no rotation. The effect of the tidal brake is that the earth never has quite the full amount of precession, or the free precession, necessary to maintain its inclination constant, and the axis slowly falls away.

We may therefore divide the motion into two parts, viz., the actual precession combined with nearly all, but not quite all, of the gravitational couple, resulting in a precessional motion with constant inclination, together with an extremely minute gravitational couple which is unbalanced by any precession and which results in a pendulation through the pole of the attracting orbit, exactly as if there were no rotation.

Regarding it from another point of view, we may consider the actual precession to be a free precession together with a minute precession in the opposite, or positive, direction, the algebraic sum of the two being the actual precession. The motion can thus be divided into two parts — the free precession which exactly balances the gravitational couple and which would maintain the inclination with no other forces, together with a minute

direct, or positive, precession unbalanced by any gravitational couple, which causes the axis to pendulate through the pole of the ecliptic, exactly as in the case of the gyroscopic compass. Whether regarded as an ordinary gravitational pendulum, or as a gyroscopic pendulum, the motions are equivalent.

Considering the earth as absolutely rigid, its axis would precess in a small circle about the pole of the ecliptic at a constant average inclination forever. But the sweep of the tide is equivalent to a minute positive couple about an axis $\perp$ to the ecliptic, with the result that the precessional curve is not exactly re-entrant but gradually spirals in towards the pole.

Let us consider the following problem which is similar to, but not identical with, the actual case. We shall consider the earth to be absolutely rigid (non-deformable), and a constant positive couple, $H$, is applied about an axis through its centre perpendicular to its orbit. Let $\omega$ be the rotational velocity of the earth at the beginning of an epoch and $\omega_3$ that at any subsequent time. The average gravitational couple for a complete revolution we have found to be $-\frac{K}{2}\sin\vartheta\cos\vartheta$, or half the maximum couple. We can effect the same average precessional motion by placing the half mass of the attracting body at the pole of its orbit at a distance equal to its average distance, and supposing it to repel instead of attracting.

The equations of motion are,

$$-\frac{K}{2}\sin\vartheta\cos\vartheta - C\omega_3\dot{\psi}\sin\vartheta + A\dot{\psi}^2\sin\vartheta\cos\vartheta = A\ddot{\vartheta} \quad (1)$$

$$H\sin\vartheta + C\omega_3\dot{\vartheta} - A\dot{\psi}\cos\vartheta\dot{\vartheta} = AD_t(\dot{\psi}\sin\vartheta) \quad (2)$$

$$H\cos\vartheta = CD_t\omega_3 \quad (3)$$

From (2) and (3) we derive the momental equation,

$$Ht = C\omega_3\cos\vartheta - C\omega\cos\vartheta_0 + A\dot{\psi}\sin^2\vartheta \quad (4),$$

which states that the increase of the moment of momentum about the axis of the couple is measured by the time integral of the couple. From (4),

$$A \int \dot{\psi} \sin^2 \vartheta dt = \frac{Ht^2}{2} - \frac{C^2}{2H}(\omega_3^2 - \omega^2) + C\omega \cos \vartheta_0 t,$$

since $$\int C\omega_3 \cos \vartheta dt = \frac{C^2}{2H}(\omega_3^2 - \omega^2).$$

From (1) and (2) we have the energy equation,

$$\frac{K}{4}(\sin^2 \vartheta_0 - \sin^2 \vartheta) + \frac{H^2t^2}{2A} - \frac{C^2}{2A}(\omega_3^2 - \omega^2) + \frac{C\omega \cos \vartheta_0 t}{A} H = \frac{A}{2}(\dot{\vartheta}^2 + (\dot{\psi} \sin \vartheta)^2). \qquad (5).$$

It is evident that the axis spirals in towards the pole with a negative precession and at the pole, $Ht = C(\omega_3 - \omega \cos \vartheta_0)$, while the value of $\dot{\vartheta}^2$ at the pole is

$$\dot{\vartheta}^2 = \frac{K}{2A}\sin^2 \vartheta_0 + \frac{C^2}{A^2}\omega^2 \sin^2 \vartheta_0.$$

Since $K$ is small compared with $\omega$, we may write

$$\dot{\vartheta} = \frac{C}{A}\omega \sin \vartheta_0.$$

Thus the polar value of $\dot{\vartheta}$ depends only upon $\omega$ and is independent of any intermediate values, $\omega_3$, of the rotational velocity. If there were a couple retarding the rotational velocity, as in the actual case, instead of an accelerating couple, the polar value of $\dot{\vartheta}$ would be the same.

We shall now start from the pole, as a new epoch, with a velocity,

$$\dot{\vartheta} = \frac{C}{A}\omega \sin \vartheta_0.$$

Let $\omega_p$ be the rotational velocity at the beginning of this epoch. The momental equation is now

$$Ht = C\omega_3 \cos \vartheta - C\omega_p + A\dot{\psi} \sin^2 \vartheta \qquad (6).$$

It is evident that as the axis spirals outwards the precession will be direct, or in a positive direction. Equa. (5) now becomes

$$-\frac{K}{4}\sin^2 \vartheta + \frac{H^2t^2}{2A} - \frac{C^2}{2A}(\omega_3^2 - \omega_p^2) + \frac{C\omega_p Ht}{A} = \frac{A}{2}\left(\dot{\vartheta}^2 - \frac{C^2\omega^2}{A^2}\sin^2 \vartheta_0\right) + \frac{A}{2}(\dot{\psi} \sin \vartheta)^2. \qquad (7).$$

Putting the value of $Ht$ from (6) in (7) and making $\dot{\vartheta} = 0$,

$$A\dot{\psi} \sin \vartheta \cos \vartheta = C\omega_3 \sin \vartheta \pm \left[ C^2\omega^2 \sin^2 \vartheta_0 - \frac{KA}{2} \sin^2\vartheta \right]^{\frac{1}{2}} \quad (8).$$

Since $K$ is small compared with $\omega$, we can write

$$A\dot{\psi} \sin \vartheta \cos \vartheta = C\omega_3 \sin \vartheta \pm C\omega \sin \vartheta_0. \quad (9).$$

This is the condition for the extreme outward swing. It is evident that the precession will still be positive when $\dot{\vartheta}$ becomes zero. $\dot{\psi} \sin \vartheta \cos \vartheta$ and $\omega_3 \sin \vartheta$ are the components in the orbital plane of the rotational velocities about their respective axes. These components are about the same axis but in opposite directions and therefore have different signs. If we consider $\dot{\psi} \sin \vartheta \cos \vartheta$ negative, then $\omega_3 \sin \vartheta$ is positive and for the right member of (9) to be negative we must use the lower sign and $C\omega_3 \sin \vartheta < C\omega \sin \vartheta_0$. If we consider $\dot{\psi} \sin \vartheta \cos \vartheta$ positive, then $C\omega_3 \sin \vartheta$ is negative and for the right member to be positive, we must use the upper sign and $C\omega_3 \sin \vartheta < C\omega \sin \vartheta_0$. But $\omega_3 > \omega$, whence $\sin \vartheta < \sin \vartheta_0$. Consequently the axis starts on its second swing towards the pole from a nearer position than on the first swing. It will be noted that in Equa. (9) the rotational velocity $\omega_p$ for the beginning of the epoch does not appear. Consequently if during the outward swing the rotational velocity were retarded, as is actually the case, instead of being accelerated, the motion would be similar. Any variation of the velocity about the polar axis does not influence the direction or general nature of the motion about the other axes: it merely modifies slightly the amount of such motions.

Making $\dot{\psi} \sin \vartheta = 0$, we have

$$\dot{\vartheta} = -\frac{C}{A}\left[ (\omega^2 \sin^2 \vartheta_0 - \omega_3^2 \sin^2 \vartheta) \right]^{\frac{1}{2}}.$$

The point where the precession becomes retrograde is therefore within the original starting point and it is evident that the axis will finally come to rest $\perp$ to the orbit.

The motion is represented diagrammatically in Fig. 25. The full curve represents the retrograde spiral inward and the dotted curve the direct spiral outward. At $A$ the

axis has ceased going outward and at $B$ the precession becomes retrograde. Each inward spiral is begun successively nearer to the pole. If the couple, $H$, is very small, the period of each swing is very great.

The actual case of the earth, while generally similar, differs in some respects from the preceding problem. The diurnal rotation, instead of being accelerated, is retarded, but the precessional rotation is accelerated, as in the problem. We do not know the exact expressions for the couples about the two axes, and if we did the equations of motion would probably not be integrable. However, the general nature of the motion is evident and it must be like that in the preceding problem. In Equa. (1) the factors governing the polar motion of the axis are the gravitational couple and the gyroscopic couple, $-C\omega_3\dot{\psi}\sin\vartheta$. When the precessional and diurnal motions are both accelerated it is evident that the axis gets nearer to the pole with every swing. However, whether the amplitudes of the swings successively decrease or increase depends upon the relative variation of the two factors, $\omega_3$ and $\dot{\psi}\sin\vartheta$, — the rotations about the two axes. If $\omega_3$ decreases in a relatively greater proportion than the precession is accelerated, the amplitude of each swing will become progressively greater until the positive end of the axis of a planet may be brought to the other side of the ecliptic, and its diurnal rotation will appear to be retrograde.

FIG. 25.

We may consider the nearer satellites of a planet as rigidly attached, dynamically, to its equatorial plane, so that the orbits of the nearer satellites will be "tipped over" with the planet, and their revolutions will appear to be retrograde. The fact that such motions occur in our solar

system is therefore not an argument against the nebular hypothesis. The less the density and rigidity of a planet, the more likely is it that its diurnal rotation will be slowed down disproportionately to the tidal acceleration of its precession, resulting in an extreme pendulation. If, in the case of the gyroscopic compass, the precession remains practically constant while the rotational velocity steadily decreases, an extreme pendulation will result, even through the opposite pole.

The problem as treated here does not take account of the fact that besides the breaking action of the tide, the earth is distorted, so that the tidal mass is actually performing an independent rotation in the plane of the ecliptic. This gives rise to a moment of momentum about the axis of the ecliptic equal to the mass of the tide, into the square of its distance from the centre, into the orbital velocity. This is the same as if the earth were perfectly rigid and had an added moment about the orbital axis converting it into a gyroscopic compass. The effect is very slight, but additive to the breaking effect.

If the earth were a perfectly rigid body, it would maintain its inclination constant, but perfectly rigid bodies do not exist. The motion of an elastic body is different and, given sufficient time, the results are widely different. There is not the least doubt that the earth executes pendulations through the pole of the ecliptic, and there is abundant evidence that it has executed several such swings in the past. The extent of glaciation about the poles at any time is simply a function of the axial inclination. With the axis $\perp$ to the ecliptic there would be no ice anywhere and a genial climate would exist even at the poles. There is undoubted evidence that a subtropical flora flourished near the poles in a comparatively recent past. This is positive proof that the axis at that time had little inclination, for this flora could not have flourished without continuous light as well as heat. With the present inclination there are extensive ice caps at the poles and with a few degrees more

of inclination these caps would extend to twice their present area, as happened during the last extreme glaciation.

The period of the swing must be enormously great. What it is we do not know and perhaps centuries must elapse before we have any definite knowledge.

It is stated that Eratosthenes (B.C. 250) found the inclination to be 23° 51′ 20″ and that Hipparchus (B.C. 120) found it 23° 51′. The determination of this angle requires a high state of contemporaneous civilization. Only a few centuries ago such determinations were nowhere possible. Shortly before and after the Christian era there were astronomers who could attempt it. Going farther back we come to another stage of barbarism and beyond this in a very remote antiquity, we come to the Pyramid builders, who were astronomers of a high order. Their time has been estimated all the way from 3000 to 13,000 B.C. and even more, but the simple fact is we do not know when they lived.

These builders have left us a peculiar angle in their oldest pyramids. This angle is 26° and its double 52°. The slopes of all the faces are 52° and the inclination of all the passages, whether descending or ascending, is 26°. The selection of this particular angle and its constant repetition could not have been accidental. They undoubtedly had measured the inclination of the earth's axis to within a minute. This is the one great angle in all nature which would impress itself upon intelligent men. There is no other predominant angle for an earth-dweller. There is a presumption then that this angle was the inclination of the earth's axis at that time and that its double, or 52°, was the breadth of the then tropical zone. But, of course, this is only a surmise.*

**v.* Popular Astronomy. Dec. 1916.

### 37. The Earth's Surface

The rigid demonstration of a mathematical proposition is something which must be eternally true. There are, however, many cases where we cannot hope to arrive at exact results, but only at probable truths, or even at possible truths. It is thus legitimate to speculate, provided we always keep carefully in mind the distinction between exact knowledge and surmise. In the present article we shall allow ourselves to speculate upon what part known forces *may* have had in shaping the earth's surface.

The earth is progressively denser from the surface to the centre, and layers of equal density are ellipsoids. The ellipticity, or deviation from sphericity, of these surfaces increases from the centre, where it is zero, to the surface. The principal moments of inertia of these shells vary therefore, and the quantity, $C - A$, upon which the precession depends, increases to the surface. The precessional rates of the different shells being different and the interior being plastic, the precession is not executed as a whole, as it would be in a perfectly rigid body, but there is a tendency for some of the outer shells to move over each other. Since it is not possible for a spheroidal shell to turn about its long axis over an enclosed shell very far, such motion must be very limited. There result, however, readjustments which may be smooth and regular, or occasionally effected suddenly. Thus earthquakes arise which, though actually very slight movements, seem to observers like men to be of extraordinary intensity. It is evident that if the earth were isolated such phenomena could not arise and their occurrence points plainly to the action of external bodies.

The effect of the shifting of large masses over the earth's surface is an interesting problem. The aqueous vapor in the atmosphere is a small but appreciable fraction of the earth's mass. Ordinarily the weight of the atmosphere is

distributed in a fairly even manner over the earth's surface, but if during the beginning of a glacial epoch the aqueous content going poleward is locked up there and not allowed to return, a disturbance of equilibrium results. The first result of a concentration of matter towards the poles is a diminution of the moment of inertia about the earth's axis with a corresponding increase in the rotational velocity, since the moment of momentum must remain constant. A secondary result, which is corrective of the former, is a bulging of the equatorial regions, due to the increased rotation and the increased weight at the poles, as the earth strives to regain its former potential surface. These effects are extremely slight, but an inequality may remain uncorrected for some time and a sudden readjustment may result in appreciable effects. A moderately rapid change of the rotational velocity, by a fraction of a second, would result in tangential stresses which would throw up long north-south ridges (mountain chains). A moderately rapid change of the ellipticity of the earth would result in the throwing up of east-west ridges, and as these two effects might occur simultaneously, we may have diagonal ridges. Owing to the plasticity of the earth, it keeps its surfaces, both interior and exterior, very nearly in an equipotential condition at all times. Deviations may accumulate for a short time and then be corrected suddenly (catastrophically), but the divergence is never wide. The existence of ridges in the cardinal directions and their diagonals, strikingly exhibited upon the earth's surface, points to the dynamical (rotational) causes which we have just considered.

During the times in the past also, when the axis was $\perp$ to the ecliptic, the earth must have been subjected to peculiar and violent stresses. At this point the angular velocity about the $\vartheta$ axis is a maximum, and this axis shifts suddenly in the plane of the equator to a point 180° opposite and then back again just before and just after the pole is passed. Or in a comparatively short time the

comparatively large $\dot{\vartheta}$ velocity is reversed. Such a catastrophic commotion must result in extensive fracturing of the earth's crust and the throwing up of east-west ridges.

The older ideas that the inequalities of the earth's surface were due to the adaptation of its crust to a slowly contracting (cooling) core, are found upon examination to be untenable. The effect would be too slight to produce the observed phenomena, and if there were such an effect it would be entirely different.

We may therefore provisionally conclude that seismic disturbances at the present time in all probability have their origin in the earth's precession, and that the major upheavals of the past probably were caused by minute though rapid changes in the rotational velocity, and especially at a particular time when the earth's axis was $\perp$ to the ecliptic. There was probably a connection between some of these upheavals and former glacial periods.

## 38. Sufficiency of Natural Forces

Starting with the fundamental law, $f = m\dfrac{dv}{dt}$, we have derived all the main principles of natural philosophy and explained many of the actions continually taking place about us. The mathematical, or inductive, method employed is merely a system of close and careful reasoning — the only one by which absolutely true results can be secured. Some of the proofs have been given in words but are none the less mathematical for that reason. Symbols are merely a shorthand for recording various steps in the reasoning.

Natural philosophy is not limited to the scrutiny of particular problems, but should supply us with an insight into all matters — even the highest. It does not follow that all or any of these higher problems will necessarily ever be solved, but if they ever are solved it must be by this method of strict and careful reasoning, or "organized common sense." In all our experience we have never been

able to recognize more than two things, viz., matter and motion, or simply moving matter.

The question arises whether there is something more which we have hitherto failed to recognize — a *tertium quid*. We know that life consists of moving matter and that when the motion ceases the system becomes dis-associated. Was there here something more? If there was, it was not matter and it was not force or motion, since motion exists only in connection with matter.

We have solved completely only a few of the simpler phenomena resulting from matter in motion. As we advance in this study we notice the increasing complexity of the phenomena. As the factors increase the results become bewildering and our brains which are composed of only a limited number of cells of matter in motion are unable to follow them. Reasoning inductively from the complicated phenomena which a few particles in motion can produce to what an infinity of such particles under an infinitude of forces should be able to produce, we can impose no limit to the resulting phenomena. We have no right, therefore, from our experience, to deny that there are any phenomena which cannot be effected by matter in motion. We have explored only a part of the field and beyond are vast regions which may always be beyond our reach, but as we progress there is nowhere any evidence of a limit — only an unlimited vista and increasing complexity. The mathematical theory of probability projects a certain trend into the unknown, often with surprising accuracy, and both our experience and probability point strongly to the sufficiency of matter in motion.

The tendency of mankind, on seeing phenomena which it does not understand, is to ascribe them to supernatural agencies, and this attitude has an important bearing on our present inquiry. Primitive men see in all the phenomena of nature, spirits, good and bad. In the past, as in the present, there have always existed in the minds of men hosts of elves, goblins, ghosts, daemons and what-not, who are

continually performing supernatural acts. These spirits and their acts have formed the bases of their religions and to deny that our present religions are evolved from them is simply to deny that there has been mental as well as physical evolution.

When a Samoan is photographed he believes that a daemon is in the camera and when a Hottentot hears a phonograph he has no doubt but that a spirit is producing the sounds. We have a smug way of imagining ourselves very much superior to populations of the past, but the difference is only in degree and very slight at that. We have very recently learned how to utilize a few of the forces of nature, which the average man does not understand, but on the whole it is very likely that the average ancient Egyptian was about as intelligent as the average man of today. If an advanced intellect at some time in the future shall look back upon both of us, he will probably find little difference. For the most civilized and educated among us firmly believe in miracles in the past, if not in the present, and many of our ideas, if analyzed by such an advanced intellect, would appear most extraordinary.

The average civilized man of today is in advance of the Samoan in that he does not see the necessity of having a daemon in a camera, but when it comes to a monad they are on all fours and both insist upon the daemon. The natural philosopher, with a fuller understanding of matter and motion cannot share this belief. And he has not the least desire that others should share his view unless they can recognize that the probability, amounting almost to a certainty, is that the camera and the monad are both phenomena of matter in motion. It is not impossible that we shall sometime be able to produce life artificially.

The vexed question of the immortality of the soul is a simple one for the natural philosopher. Under all arguments for such an immortality there stands out plainly the personal desire of the pleader that, having existed for some few years as a congeries of certain moving carbon,

nitrogen, oxygen and hydrogen atoms, he may somehow and in some way continue a very different kind of existence for all eternity. While all the rest of nature is continually undergoing flux and evolution, he alone remains fixed forever! There have been many who have had no desire for such an eternal existence, but they have based no arguments upon their personal wishes. Both the matter and the motion of a living organism are immortal, but they no longer form the same system after dissolution. The natural philosopher cannot hold the view that the billions of the earth's past populations — without considering the lower animals — still exist as separate entities, or disembodied souls.

It may seem that such questions are wholly foreign to our subject, but the domain of the natural philosopher is the whole universe, and there is nothing in it he may not philosophize about, provided he preserves the strict methods of his science. To many, such questions may seem to be axioms, unworthy of serious discussion. But it must be remembered that the great majority of living men firmly believe in miracles and are convinced that the laws of nature are not really laws, or at best are only laws for a part of the time, since occasionally they are broken. It is possible that in some higher stage of advancement mankind may finally use the means at his disposal for obtaining the truth and cleave to it.

# NOTE

## On the Cause of Gravitation

The ether is the seat of an enormous store of energy as evidenced by its enormous pressure. In the last analysis this energy must be kinetic, or due to some kind of motion within the ether, although we have as yet not the slightest conception of the nature of such a motion. The atoms of gross matter, being imbedded in the ether, necessarily partake of this motion just as specks within a liquid partake of the motions of the surrounding molecules, constituting the well known Brownian movements. They are thus foci which reflect and radiate the internal ethereal vibrations.

The atoms being in incessant motion and the ether possessing both elasticity and inertia, among other disturbances, longitudinal waves necessarily result.

We shall prove that such longitudinal waves necessarily cause an attractional action between all atoms of gross matter. A longitudinal wave is composed of two halves having opposite properties. In one half the medium is above its normal density and its particles are moving with or against the wave direction, while in the other half the medium is below its normal density and the particles are moving in a reversed direction. An atom swept by the wave will therefore be urged alternately towards and away from the radiating point.

Let $v$ be the average velocity of the stream in either direction, $a$ the amplitude of the wave, or the maximum distance any particle of the medium moves from its position of equilibrium, $V$ the wave velocity, $D$ the density of the medium, $P$ its pressure, $l$ a wave length, and $t$ the time of a complete vibration. The force with which such a

stream urges an atom is proportional to its velocity, to its density and to the surface which the atom opposes to the stream. Taking the cross section of the atom as unity, its mass as $M$, and the mass of an equal volume of the medium as $m$, it is evident that such a stream striking $M$ and $m$ at rest, will move them, in the time of half a wave, distances inversely as their masses, or while it moves $m$ a distance $2a$, just as it moves any other portion of the medium, it moves $M$ only $\frac{2\,am}{M}$. Or $Ms = 2am$, where $s$ is the distance $M$ is moved during a half wave. At the end of a half wave, therefore, $M$ is $2\,a\,\frac{(M - m)}{M}$ behind $m$.

The time taken by each half wave to clear $m$ is $\frac{l}{2V}$, but the time taken by the compressed half to clear $M$ is

$$\frac{l}{2V} - 2\,a\left(\frac{M - m}{MV}\right),$$

while the time taken by the expanded half to clear $M$ is

$$\frac{l}{2V} + 2\,a\left(\frac{M - m}{MV}\right).$$

The average stream pressure in either direction is the same and equal to $kDv$, $k$ depending upon the units chosen. But the time during which this pressure acts is unequal in the two halves. There is a net pressure acting for a time $4\,a\left(\frac{M - m}{MV}\right)$, in the direction of motion of the expanded half, during the passage of every complete wave. This is equivalent to a force acting continuously equal to $\frac{kDv}{t} \cdot 4\,a\left(\frac{M - m}{MV}\right)$. In all gravitational waves the expanded half moves towards the radiating source. Since $\frac{v}{V} = \frac{4\,a}{l}$, the force urging the atom contrary to the wave direction is $kD\,\frac{16\,a^2}{l} \cdot \frac{M - m}{Mt}$. The potential and

kinetic energies in a complete wave are equal, and it can be shown (*v.* "Mechanics of Electricity") that the total energy in a wave, per unit cross section, is

$$\frac{16a^2DV^2}{l} = \frac{16a^2P}{l}.$$

Calling this total wave energy, $E$, the attractive force

$$f = \frac{k}{V^2} \cdot \frac{M - m}{M} \cdot \frac{E}{t}.$$

Now $\frac{E}{t}$ is the energy crossing unit surface in unit time, or the flux of energy per unit surface. We may call it the density of the energy flux. The term "Flux of force," frequently used, is meaningless. There is no such thing as a flux of force, but a flux of energy constitutes a force.

If we take an equal volume of the medium in place of $M$, or make $M = m$, there is no force and the atom merely vibrates about its position of equilibrium. If the particle is less dense than the medium, the force becomes negative and there is a repellant instead of an attractive action. It is evident that a body less dense than the ether will be driven up very quickly to the limiting velocity, $V$, when, since it travels with the wave, all further action ceases. If $M$ is very much denser than the ether, as is the case for all gross matter, $f = \frac{k}{V^2} \cdot \frac{E}{t}$, or for all gross matter opposing unit surface to the wave, the attraction is simply proportional to the density of the energy flux.

Considering positive and negative charges of electricity as differentiated portions of the ether which are respectively denser and less dense than the normal ether, electrostatic attractions and repulsions necessarily result from the causes just discussed (*v.* "Mechanics of Electricity"). At a time when longitudinal waves in the ether were denied, as in fact they are today, Lord Kelvin wrote, "I affirm, not as a matter of religious faith, but as a matter of strong scientific probability, that such waves (compressional) exist, and that the velocity of this unknown

condensational wave is the velocity of the propagation of electrostatic force." Lord Kelvin, Baltimore Lectures.

There is no doubt that all atoms ceaselessly radiate longitudinal waves. Such being the case, universal attractions and repulsions necessarily follow. Hence, if anyone should seek to explain gravitation through some other agency, he would still have this agency unavoidably coupled with it. Nature works in the simplest way possible. She does not employ multiple agencies to produce a simple effect. Further there is no other conceivable agency by which such an action could be effected.

This action of longitudinal waves is not confined to the ether but is a property of longitudinal waves in any medium. It is readily verified experimentally in air (sound) waves. Gravitational force is the *push*, not pull, of the ether streams against the atomic surfaces, and hence is proportional to the cross section opposed. The work of some experimenters seems to show that certain atoms, or at least certain arrangements of atoms (molecules) may have different cross sections in different directions. Thus Heydweiler claims that the weight of a crystal of $CuSO_4$, where the atoms are presumably oriented, is not the same as that of the same mass in solution, where the atoms are supposed to be unoriented. Wallace claims that the weight of a given mass of water changes after it is frozen, i.e., crystallized or oriented, but as yet we have very little knowledge concerning such matters. The weight of a given mass should vary with the orientation of its atoms to the field, if the cross-sections of the atoms vary with the direction.

*END*